# CHAPTER 23

# Goodness-of-Fit Criteria Based on Observations Quantized by Hypothetical and Empirical Percentiles

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Goodness-of-fit disparity statistics are defined as appropriately scaled $\phi$-disparities or $\phi$-divergences of quantized hypothetical and empirical distributions. It is shown that the classical Pearson-type statistics are obtained if we quantize by means of hypothetical percentiles, and that new spacings-based disparity statistics are obtained if we quantize by means of empirical percentiles. The main attention is paid to the asymptotic properties of the new disparity statistics and their comparisons with the spacings-based statistics known from the literature. First the asymptotic equivalence between them is proved, and then for the new statistics a general law of large numbers is proved, as well as an asymptotic normality theorem both under local and fixed alternatives. Special attention is devoted to the limit laws for the power divergence statistics of orders $\alpha \in \mathbb{R}$. Parameters of these laws are evaluated for $\alpha \in (-1, \infty)$ in a closed form and their continuity in $\alpha$ on the subinterval $(-1/2, \infty)$ is proved. These closed form expressions are used to compare local asymptotic powers of the tests based on these statistics, which allows to extend previous asymptotic optimality results to the class of power divergence statistics. Tables of values of the asymptotic parameters are presented for selected representative orders of $\alpha > -1/2$.

*Key words:* asymptotic normality, asymptotic optimality, consistency, goodness-of-fit, power divergences, spacings, $\phi$-disparities, $\phi$-divergences.

## 1 Data and their statistical models

In this chapter we consider the explanation of *observed data* $x_1, x_2, ..., x_n$ statistically as a sequence of independent outcomes from a statistical model. Our aim is to review and extend the criteria of goodness-of-fit of the model and data, and to study their properties and applications in decisions about the acceptability of concrete models for concrete data.

Let us start with the example of data first studied by Pearson (1894), which represent measurements of the ratio of the forehead to body of $n = 1000$ crabs. Pearson partitioned the original domain of measurements $(a, b) = (0.5795, 0.6995)$ into intervals of equidistant size 0.004 and counted the frequency for each interval. Table 1.1 gives the measurement values $v_j$ represented by midpoints of the intervals and the corresponding frequencies $\varphi_j$ for $1 \le j \le 30$.

**Table 1.1** Ratio of the forehead to body of 1000 crabs

| Value | Freq. | Value | Freq. | Value | Freq. | Value | Freq. | Value | Freq. |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 0.5815 | 1 | 0.5855 | 3 | 0.5895 | 5 | 0.5935 | 2 | 0.5975 | 7 |
| 0.6015 | 10 | 0.6055 | 13 | 0.6095 | 19 | 0.6135 | 20 | 0.6175 | 25 |
| 0.6215 | 40 | 0.6255 | 31 | 0.6295 | 60 | 0.6335 | 62 | 0.6375 | 54 |
| 0.6415 | 74 | 0.6455 | 84 | 0.6495 | 86 | 0.6535 | 96 | 0.6575 | 85 |
| 0.6615 | 75 | 0.6655 | 47 | 0.6695 | 43 | 0.6735 | 24 | 0.6775 | 19 |
| 0.6815 | 9 | 0.6855 | 5 | 0.6895 | 0 | 0.6935 | 1 | 0.6975 | 0 |

As second example we use the data studied recently by Ning, Gao and Dudewicz (2008) which are results of measurements of cadmium concentrations in the kidney cortex of $n = 43$ horses. These are presented in Table 1.2 below.

**Table 1.2** Amounts of cadmium in horse kidneys

| 11.9 | 16.7 | 23.4 | 25.8 | 25.9 | 27.5 | 28.5 | 31.1 | 32.5 | 35.4 | 38.3 |
|------|------|------|------|------|------|------|------|------|------|------|
| 38.5 | 41.8 | 42.9 | 50.7 | 52.3 | 52.5 | 52.6 | 54.5 | 54.7 | 56.6 | 56.7 |
| 58.0 | 60.8 | 61.8 | 62.3 | 62.5 | 62.6 | 63.0 | 67.7 | 68.5 | 69.7 | 73.1 |
| 76.0 | 76.9 | 77.7 | 78.2 | 80.3 | 93.7 | 101.0 | 104.5 | 105.4 | 107.0 | - |

**Statistical models** are probability distributions on data spaces $\mathcal{X}$. A general probability distribution is specified by a **probability measure** (briefly, **p.m.**) $P$ defined on appropriate subsets of $\mathcal{X}$. The data spaces of the examples in Tables 1.1 and 1.2 are the halflines $(0, \infty)$. Throughout this chapter we restrict ourselves to real valued data with the data space being an interval $\mathcal{X} = (a, b) \subseteq \mathbb{R}$. Then each probability measure $P$ is uniquely specified by the **distribution function** (briefly, **d.f.**)

$$F(x) = P((-\infty, x]), \quad x \in \mathbb{R}$$

We restrict ourselves to statistical models with increasing and continuously differentiable d.f.'s $F(x)$ on the data space . These are uniquely specified by the positive continuous **probability density functions** (briefly, **p.d.f.**'s)

$$f(x) = \frac{\mathrm{d}F(x)}{\mathrm{d}x} \quad x \in (a, b)$$

as well as by the increasing **percentile functions** (briefly, **p.f.**'s)

$$Q(y) = F^{-1}(y) \quad (\text{i.e. } x \in [a, b] \text{ such that } F(x) = y), \quad y \in [0, 1]$$

where

$$Q(0) = a \quad \text{and} \quad Q(1) = b$$

due to the assumption that $F(x)$ is increasing on $(a, b)$. Therefore the observed data $x_1, x_2, ..., x_n$ are interpreted as realizations of independent copies $X_1, X_2, ..., X_n$ of a random variable $X$ which can be specified equivalently by a p.m. $P$ on $\mathcal{X}$ or a d.f. $F(x)$ on $\mathbb{R}$ or a p.d.f. $f(x)$ on the data space $(a, b)$, or by a p.f. $Q(y)$ on the percentile space $[0, 1]$.

The data $x_1, x_2, ..., x_n$ themselves can be represented by the so-called **empirical probability measure** (briefly, **e.p.m.**)

$$P_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i} \tag{1.1}$$

on the data space $\mathcal{X} = (a, b)$ where $\delta$ denotes the Dirac probability measure, or by the related **empirical distribution function** (briefly, **e.d.f.**)

$$F_n(x) = P_n((-\infty, x)) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{I}(x \geq x_i) \tag{1.2}$$

on $\mathbb{R}$ where $\mathbf{I}$ denotes the indicator function. Both these representations are unique up to the ordering of the data. In other words, $P_n$ as well as $F_n$ are one-to-one related to the order statistic $(x_{n:1}, x_{n:2}, ..., x_{n:n})$ of the data vector $(x_1, x_2, ..., x_n)$. The loss of order means no loss of statistical information, because the order statistics are known to be statistically sufficient for the statistical models independent identically distributed (i.i.d.) observations (cf. Lehmann and Romano (2005)).

An alternative form of presentation of the data $x_1, x_2, ..., x_n$ is by the histogram $f_n(x)$ on the data space $(a, b)$. This is the density of the restriction of the e.p.m. $P_n$ on the algebra generated by the $k + 1$ intervals obtained by slicing $(a, b)$ at the cutpoints

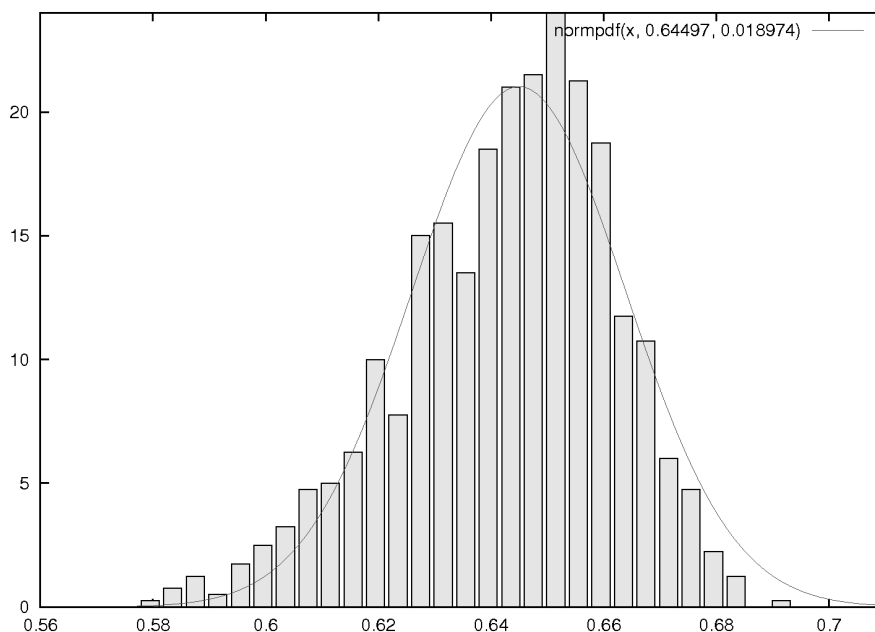$$a = c_0 < c_1 < \ ... \ < c_k < c_{k+1} = b.$$

Figure 1.1: Normal approximation of the crab data from Table 1.1.

Then for every $x \in [c_j, c_{j+1})$

$$f_n(x) = \#x_i \in [c_j, c_{j+1}).$$

For example, the frequencies $\varphi_j$ of Table 1.1 define the histogram

$$f_n(x) = \varphi_j \quad \text{for} \quad x \in [c_j, c_{j+1}) \tag{1.3}$$

where

$$c_j = 0.5795 + 0.004 \cdot j \quad \text{for} \quad 0 \le j \le 30 \tag{1.4}$$

are uniform cutpoints of the interval $(a, b) = (0.5795, 0.6995)$. The histogram form of presentation of data is statistically sufficient in exceptional situations only.

Figure 1.1 compares the histogram (1.3) presenting the data from Table 1.1 with the p.d.f. $f(x)$ of the maximum likely normal model $N(\mu, \sigma^2)$ of these data where

$$\mu = \frac{1}{1000} \sum_{i=1}^{1000} x_i = 0.6447 \quad \text{and} \quad \sigma^2 = \frac{1}{999} \sum_{i=1}^{1000} (x_i - \mu)^2 = 0.00036. \tag{1.5}$$

Figure 1.2 shows the e.d.f. $F_n(x)$ of the data from Table 1.2 together with the d.f. $F(x)$ of the normal model $N(\mu, \sigma^2)$ with sample mean and variance

$$\mu = \frac{1}{43} \sum_{i=1}^{43} x_i = 57.2326 \quad \text{and} \quad \sigma^2 = \frac{1}{42} \sum_{i=1}^{43} (x_i - \mu)^2 = 574.98. \tag{1.6}$$

In this chapter we deal with quality criteria for fitting various possible statistical models $F(x)$ to the data represented by e.d.f.'s $F_n(x)$ including the asymptotic properties of these criteria for $n \to \infty$. The basic concepts and notations introduced in this section are used throughout all what follows below.
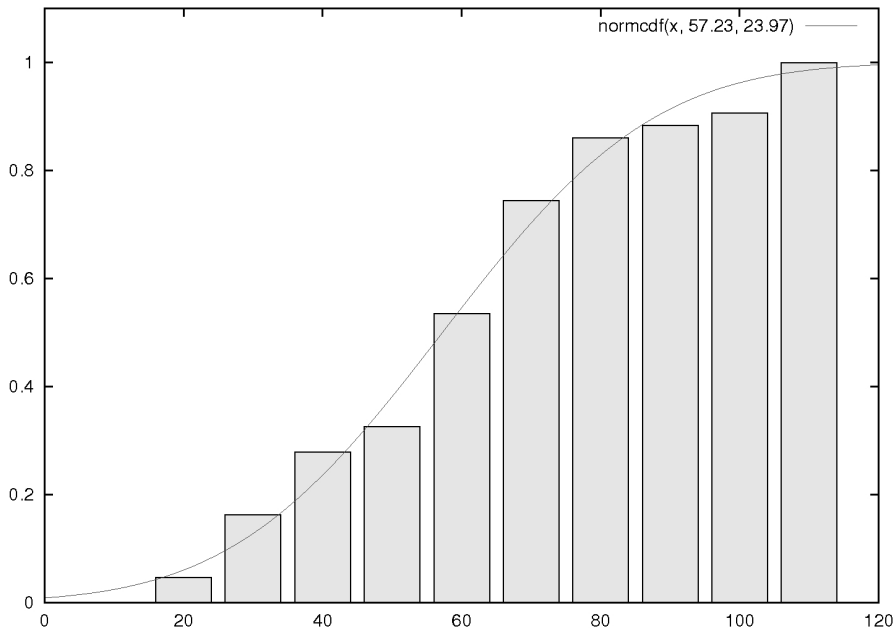
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Figure 1.2: Normal approximation of the horse data from Table 1.2.

# 2 Assessment of goodness-of-fit

Intuitively one can expect that all numerical goodness-of-fit criteria will be measures of distance, divergence or disparity between on the one hand the empirical reality represented by a data-based p.m. $P$ or d.f. $F$, e.g. the e.p.m. $P_n$ or e.d.f. $F_n$, and on the other the hypothetical statistical model given by a p.m. $P_0$ or a d.f. $F_0$. In this chapter the terms distance, divergence and disparity have a specific mathematical meaning which is specified in the definitions below, where we deal primarily with p.m.'s $P, P_0$ rather than with d.f.'s $F, F_0$.

**Definition 2.1** **(i)** By a *distance* $D(P, P_0)$ of p.m.'s $P, P_0$ we mean a standard mathematical metric distance, taking values in the interval $[0, \infty)$, which is reflexive (i.e. $D(P, P_0) = 0$ if and only if $P = P_0$), symmetric (i.e. $D(P, P_0) = D(P_0, P)$) and satisfies the triangle inequality (i.e. $D(P, P_0) \leq D(P, P_1) + D(P_1, P_0)$ for all p.m.'s $P, P_0, P_1$).

**(ii)** A **divergence** (more precisely, an **information-theoretic divergence**) $D(P, P_0)$ is a reflexive functional taking values in the interval $[0, \infty)$ and satisfying the information processing law. To formulate this law, let $T : \mathcal{X} \to \mathcal{Y}$ be a mapping describing the results of processing the data $x$ from the observation space $\mathcal{X}$ towards another space $\mathcal{Y}$, with $y = T(x)$. The information processing law says that no processing rule $T$ can increase the divergence, in symbols

$$D(PT^{-1}, P_0 T^{-1}) \leq D(P, P_0), \tag{2.1}$$

with the equality being valid if and only if $T$ is statistically sufficient for $P, P_0$.
**(iii)** A **disparity** of p.m.'s $P, P_0$ is a nonnegative functional $D(P, P_0)$ which is reflexive in the above sense.

**Convention 2.1** Throughout the chapter we denote by $P, P_0$ an arbitrary pair of probability measures on a general observation space $\mathcal{X}$. They will be represented by means of their p.d.f.'s

$$p = \frac{\mathrm{d}P}{\mathrm{d}\lambda} \quad \text{and} \quad p_0 = \frac{\mathrm{d}P_0}{\mathrm{d}\lambda} \tag{2.2}$$

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with respect to (w.r.t.) a dominating measure $\lambda$ on $\mathcal{X}$ (in symbols, $P \sim p, P_0 \sim p_0$). The only restriction imposed on the hypothetical model $P_0$ is the positivity of $p_0$ almost everywhere w.r.t. $\lambda$.

**Continuous case.** If $\mathcal{X} = (a, b) \subseteq \mathbb{R}$ then the p.m.'s $(P, P_0)$ are in a one-to-one manner represented by the d.f.'s $(F, F_0)$, and for absolutely continuous $F, F_0$ and Lebesque measure $\lambda$ it holds that

$$p = f \quad \text{and} \quad p_0 = f_0 \tag{2.3}$$

where $(f, f_0)$ are the usual p.d.f.'s of $(F, F_0)$ (in symbols, $F \sim f, F_0 \sim f_0$). As stated above, we assume $f_0 > 0$ on $(a, b)$.

**Discrete case.** If $\mathcal{X} = (1, 2, ..., k)$, then the densities $(p, p_0)$ of $(P, P_0)$ w.r.t. the counting measure $\lambda(1) = \lambda(2) = ... = \lambda(k) = 1$ reduce to the stochastic vectors

$$\boldsymbol{p} = (p_j := P(j) : 1 \leq j \leq k), \quad \boldsymbol{p}_0 = (p_{0j} := P_0(j) : 1 \leq j \leq k), \tag{2.4}$$

(in symbols, $P \sim \boldsymbol{p}, P_0 \sim \boldsymbol{p}_0$). As stated above, we assume $p_{0j} > 0$ for all $1 \leq j \leq k$.

## 2.1   Special distances, divergences and disparities

**(a) The $L_1$-distance**

$$L_1(P, P_0) = \int |p - p_0| \mathrm{d}\lambda \quad \text{for } P \sim p, P_0 \sim p_0 \tag{2.5}$$

is an example of a distance on the class of all p.m.'s which satisfies only partly the information processing law. Namely it satisfies inequality (2.1) but fails to satisfy the necessary condition for equality. To see this it suffices to consider the discrete p.m.'s $P \sim \boldsymbol{p}$ and $P_0 \sim \boldsymbol{p}_0$ given by (2.4), where

$$L_1(P, P_0) = L_1(\boldsymbol{p}, \boldsymbol{p}_0) = ||\boldsymbol{p} - \boldsymbol{p}_0||_1 = \sum_{j=1}^{k} |p_j - p_{0j}|. \tag{2.6}$$

**Example 2.1**  The binary coding $T : \mathcal{X} \to \mathcal{Y} = \{1, 0\}$ of the ternary observations $x \in \mathcal{X} = \{1, 2, 3\}$ defined by

$$T(1) = 1 \quad \text{and} \quad T(2) = T(3) = 0 \tag{2.7}$$

significantly reduces the information for discrimination between the discrete models

$$P = (1/10, \ 5/10, \ 4/10) \quad \text{and} \quad P_0 = (9/10, 1/10, 0).$$

Indeed, the discrimination rule

$$\delta(1) = P_0 \quad \text{and} \quad \delta(2) = \delta(3) = P$$

based on the original uncoded observations from $\mathcal{X} = \{1, 2, 3\}$ is errorless if $x = 3$ and the discrimination errors for the remaining observations $x$ take place with the probability $1/10$. On the other hand, arbitrary discrimination rule $\delta : \mathcal{Y} \to \{P, P_0\}$ based on the encoded data from $\mathcal{Y} = \{1, 0\}$ admits discrimination errors with the probability $9/10$. This reduction of discernibility is caused by the loss of information due to the coding which is evidently not a statistically sufficient transformation. However, this evidence is not reflected by the $L_1$- distance which remains unaffected by the coding, namely

$$L_1((1/10, \ 5/10, \ 4/10), (9/10, 1/10, 0)) = L_1((1/10, \ 9/10), (9/10, 1/10)) = 8/5.$$

Note that, nevertheless, the $L_1$-distance $L_1(f_n, f)$ is widely used as a goodness-of-fit criterion between the model p.d.f.'s $f$ and histogram-like representations $f_n$ of the observed data, since being introduced to the nonparametric statistics by Devroye and Györfi (1985).

**(b)** The $L_2$-**distance**

$$L_2(P, P_0) = \left( \int (p - p_0)^2 \mathrm{d}\lambda \right)^{1/2} \quad \text{for } P \sim p, P_0 \sim p_0$$

does not satisfy the information processing law in the sense that processing of the observations can increase the $L_2$-distance between the models $P, P_0$. To this end it suffices to consider the discrete p.m.'s $P \sim \boldsymbol{p}$ and $P_0 \sim \boldsymbol{p}_0$ given by (2.4) where

$$L_2(P, P_0) = L_2(\boldsymbol{p}, \boldsymbol{p}_0) = ||\boldsymbol{p} - \boldsymbol{p}_0||_2 = \left( \sum_{j=1}^{k} (p_j - p_{0j})^2 \right)^{1/2}. \tag{2.8}$$

The mentioned violation of the information processing law can be verified by taking $k = 3$ and the models $P \sim \boldsymbol{p} = (0,\ 1/2,\ 1/2)$ and $P_0 \sim \boldsymbol{p}_0 = (1/2,\ 1/4,\ 1/4)$ on the observation space $\mathcal{X} = \{1, 2, 3\}$. Applying the coding (2.7) to the observations $x \in \mathcal{X}$ we obtain

$$L_2(PT^{-1}, P_0 T^{-1}) = \sqrt{1/2} > L_2(P, P_0) = \sqrt{3/8}.$$

This flaw of the $L_2$- distance justifies the preference of the above-mentioned $L_1$- distance method over the $L_2$-method and underlines the importance of the $L_1$- method in statistical research.

**(c)** The **Kolmogorov distance**

$$K(F, F_0) = \sup_{x \in \mathbb{R}} |F(x) - F_0(x)| \tag{2.9}$$

introduced by Kolmogorov (1933) is the distance in the above stated metric sense and defines the well known **Kolmogorov-Smirnov goodness-of-fit statistic** $T_n = \sqrt{n} K(F_n, F_0)$ (Smirnov (1948); for more see Lehmann and Romano (2005)). Applicability of the Kolmogorov distance is restricted by the fact that it contradicts the information processing law in a similar way as the $L_2$-distance. A simple example where

$$K(FT^{-1}, F_0 T^{-1}) = 1 > K(F, F_0) = 1/2$$

is obtained by using the p.d.f.'s

$$\begin{aligned} f(x) &= \boldsymbol{I}(-1 < x < -1/2) + \boldsymbol{I}(0 < x < 1/2), \tag{2.10} \\ f_0(x) &= \boldsymbol{I}(-1/2 < x < 0) + \boldsymbol{I}(1/2 < x < 1) \tag{2.11} \end{aligned}$$

and the data processing formula

$$T(x) = x + \frac{1}{2} \left[ \boldsymbol{I}(-1/2 < x < 0) - \boldsymbol{I}(0 < x < 1/2) \right] \tag{2.12}$$

which transforms the interval $(-1/2, 0)$ on $(0, 1/2)$ and vice versa. This formula is skew symmetric about $x = 0$ and thus the data processing is reversible in the sense $T^{-1} = T$. The d.f.'s $F, F_0$ for this example are shown in Figure 2.3 and their modifications resulting from the data processing (2.12) in Figure 2.4.

For the special discrete binary p.m.'s

$$P_x \sim \boldsymbol{p}_x = (F(x), 1 - F(x)) \quad \text{and} \quad P_{0x} \sim \boldsymbol{p}_{0x} = (F_0(x), 1 - F_0(x)) \tag{2.13}$$

(2.9) and (2.6) imply

$$K(F, F_0) = \frac{1}{2} \sup_{x \in \mathbb{R}} L_1(P_x, P_{0x}).$$

Similar relations take place for other measures of goodness-of-fit between $F, F_0$ and their one-point approximations $F(x), F_0(x)$ formally represented by the binary p.m.'s (2.13). We mention the best known of them.
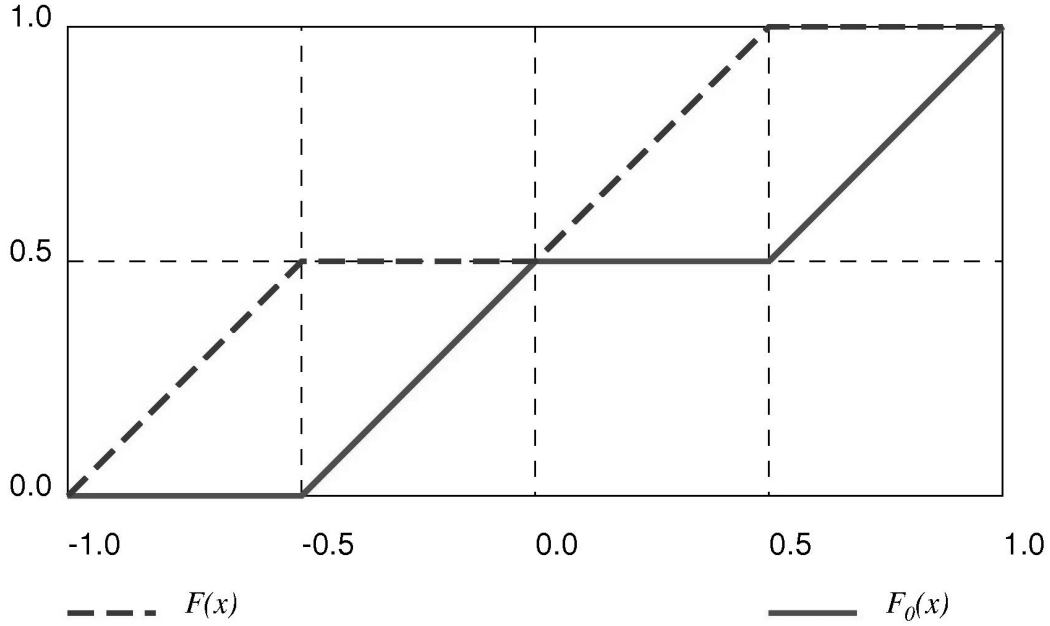
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Figure 2.3: Distribution functions $F$, $F_0$.

**(d)** The **Pearson divergence** is given by the formula

$$\chi^2(P, P_0) = \int \frac{(p - p_0)^2}{p_0} \mathrm{d}\lambda \quad \text{for } P \sim p, P_0 \sim p_0 \tag{2.14}$$

which in the discrete case considered in (2.4) reduces to

$$\chi^2(\boldsymbol{p}, \boldsymbol{p}_0) = \sum_{j=1}^{k} \frac{(p_j - p_{0j})^2}{p_{0j}}. \tag{2.15}$$

It defines the well-known **Pearson $\chi^2$ goodness-of-fit test statistic**

$$T_n = n\chi^2(\boldsymbol{p}_n, \boldsymbol{p}_0) = \sum_{j=1}^{k} \frac{(np_{nj} - np_{0j})^2}{np_{0j}} = \sum_{j=1}^{k} \frac{(\varphi_{nj} - np_{0j})^2}{np_{0j}} \tag{2.16}$$

for testing hypotheses $\mathcal{H}_0 : P_0$ on the basis of observations represented by the e.p.m. $P_n$ where $\boldsymbol{p}_0$ and $\boldsymbol{p}_n$ are the restrictions of the hypothetical and empirical p.m.'s $P_0$ and $P_n$ on the test cells $C_j = [c_j, c_{j+1})$ and $\varphi_{nj} = \#x_i \in C_j$ are the observed cell frequencies for $1 \leq j \leq k$. The Pearson divergence is a divergence in the rigorous sense stated above, but it is not a distance since it is neither symmetric nor satisfies the triangle inequality. Of course, the same holds for the **reversed Pearson divergence**

$$R\chi^2(P, P_0) := \chi^2(P_0, P). \tag{2.17}$$

**(e)** Another example of a divergence is the **LeCam divergence**

$$LC(P, P_0) = \int \frac{(p - p_0)^2}{p + p_0} \mathrm{d}\lambda \quad \text{for } P \sim p, P_0 \sim p_0 \tag{2.18}$$

which in the discrete case considered in (2.4) reduces to

$$LC(\boldsymbol{p}, \boldsymbol{p}_0) = \sum_{j=1}^{k} \frac{(p_j - p_{0j})^2}{p_j + p_{0j}}. \tag{2.19}$$
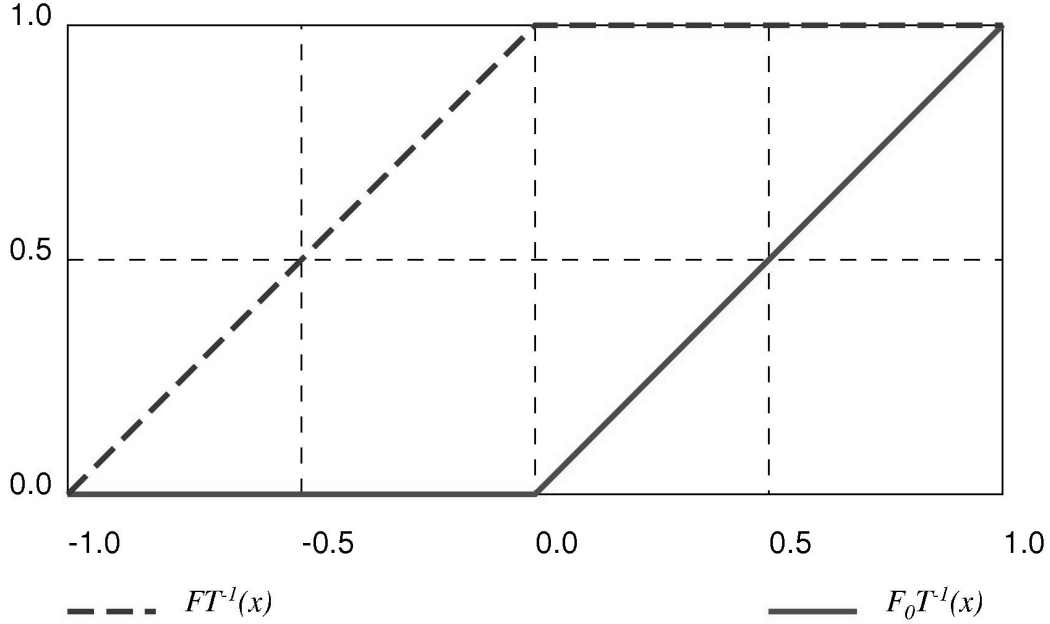
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Figure 2.4: Distribution functions $FT^{-1}$, $F_0 T^{-1}$.

Since it is a divergence in the sense of Definition 2.1, all its roots are divergences in the sense of the same definition too but, as proved by Kafka, Österreicher and Vincze (1991), the square root $\sqrt{LC(P, P_0)}$ is distinguished by being a metric distance in the space of p.m.'s $P, P_0$.

(f) In the discrete case considered in (2.13) relations (2.15) and (2.13) imply

$$\chi^2(P_x, P_{0x}) = \frac{(F(x) - F_0(x))^2}{F_0(x)} + \frac{(F(x) - F_0(x))^2}{1 - F_0(x)} = \frac{(F(x) - F_0(x))^2}{F_0(x)(1 - F_0(x))}.$$

Since $\chi^2(P_x, P_{0x})$ is a disparity of binary distributions $P_x, P_{0x}$ for all $x \in \mathbb{R}$, the integral over $\mathbb{R}$,

$$AD(F, F_0) = \int \chi^2(P_x, P_{0x}) \mathrm{d}x = \int \frac{(F(x) - F_0(x))^2}{F_0(x)(1 - F_0(x))} \mathrm{d}x, \tag{2.20}$$

is a disparity of d.f.'s $F, F_0$ in the above stated sense. We call it the **Anderson-Darling disparity** because its scaled version

$$T_n = n \cdot AD(F_n, F_0) = n \int \frac{(F_n(x) - F_0(x))^2}{F_0(x)(1 - F_0(x))} \mathrm{d}x \tag{2.21}$$

is the well known **Anderson-Darling goodness-of-fit statistic** for testing the hypothesis that the data represented by the e.d.f. $F_n$ were generated by the d.f. $F_0$ (hypothesis $\mathcal{H}_0$).

(g) Similarly, we call

$$CM(F, F_0) = \int \chi^2(P_x, P_{0x}) F_0(x)(1 - F_0(x)) \mathrm{d}x = \int (F(x) - F_0(x))^2 \mathrm{d}x \tag{2.22}$$

the **Cramér-von Mises disparity** because it is a disparity of p.d.f.'s $F, F_0$ in the above defined sense and

$$T_n = n \cdot CM(F_n, F_0) = n \int_{\mathcal{X}} (F_n(x) - F_0(x))^2 \mathrm{d}x \tag{2.23}$$

is the **Cramér-von Mises goodness-of-fit statistic** for testing the hypothesis $\mathcal{H}_0 = F_0$ under the empirical evidence $F_n$.

The goodness-of-fit statistics mentioned in **(f)** and **(g)** were introduced by von Mises (1947) and Anderson and Darling (1954) (see also Darling (1957)). We refer in this respect to pp. 58–64 in Serfling (1980) or to D'Agostino and Stephens (1986).

## 2.2 Examples

The divergences and disparities will be more systematically studied in the next section. It the remainder of this section we apply the goodness-of-fit criteria introduced in this section to the data from Tables 1.1 and 1.2.

**Example 2.2.1** Consider the discrete e.p.m. $\boldsymbol{p}_n = (p_{n1}, ..., p_{n,30})$ representing the data given by the frequencies $\varphi_j$ of Table 1.1 and defined by the formula

$$p_{nj} = \varphi_j 10^{-3}, \qquad 1 \leq j \leq 30. \tag{2.24}$$

In addition to the normal model $N(\mu, \sigma^2)$ for these data given by (1.5), we consider the mixed normal model

$$\text{MixN}(\pi, \mu_1, \mu_2, \sigma_1^2, \sigma_2^2) = \pi N(\mu_1, \sigma_1^2) + (1 - \pi)N(\mu_2, \sigma_2^2)$$

for the parameters

$$\pi = 0.5, (\mu_1, \sigma_1^2) = (0.6343, 0.000361), (\mu_2, \sigma_2^2) = (.6551, 0.00014641)$$

used by Pearson (1894). The third model considered by us is the mixed generalized lambda model

$$\text{MixGLD}(\pi, \theta, \vartheta) = \pi \text{GLD}(\theta) + (1 - \pi)\text{GLD}(\vartheta)$$

from p. 88 of Ning, Gao and Dudewicz (2008) where the generalized lambda component models are given by the percentile functions

$$Q_\theta(y) = \theta_1 + \frac{y^{\theta_3} - (1 - y)^{\theta_4}}{\theta_2} \;\; \text{and} \;\; Q_\vartheta(y) = \vartheta_1 + \frac{y^{\vartheta_3} - (1 - y)^{\vartheta_4}}{\vartheta_2}.$$

for the parameters

$$
\begin{aligned}
\pi &= 0.802 \\
\theta &= (0.6415, 13.218, 0.135, 0.205) \\
\vartheta &= (0.6564, 11.328, 0.55, 0.15).
\end{aligned}
\tag{2.25}
$$

Each d.f. $F_0 \in \{N(\mu, \sigma^2), \text{MixN}(\pi, \mu_1, \mu_2, \sigma_1^2, \sigma_2^2), \text{MixGLD}(\pi, \theta, \vartheta)\}$ represents a different hypothesis about the stochastic source of the data from Table 1.1. Each one defines a hypothetical p.d.f. $f_0$ and a hypothetical discrete p.m. $\boldsymbol{p}_0 = (p_{01}, ..., p_{0k})$ obtained by quantization of the observation space $\mathcal{X} = (a, b) \subseteq \mathbb{R}$ into $k$ cells. In the present example $(a, b) = (0.5795, 0.6995)$ and we consider quantization into $k = 30$ partition intervals symmetric about the centers $v_j$ given in Table 1.1 leding to $\boldsymbol{p}_0 = (p_{01}, ..., p_{0,30})$ with the components

$$p_{0j} = \int_{v_j - 0.002}^{v_j + 0.002} f_0(x)\mathrm{d}x = F_0(v_j + 0.002) - F_0(v_j - 0.002). \tag{2.26}$$

Table 2.2.1 presents the values of the distance or divergence criteria $L_1(\boldsymbol{p}, \boldsymbol{p}_0), L_2(\boldsymbol{p}, \boldsymbol{p}_0), \chi^2(\boldsymbol{p}, \boldsymbol{p}_0)$ and $LC(\boldsymbol{p}, \boldsymbol{p}_0)$ given by (2.6), (2.8), (2.15) and (2.19) for the e.p.m. $\boldsymbol{p} = \boldsymbol{p}_n$ given by (2.24) and hypothetical p.m.'s $\boldsymbol{p}_0$ given by (2.26). The estimate mixGLD was evaluated by the package GLDEX of the program R (project CRAN) by Steve Su. For description of the method and further citations we refer to *http://cran.r-project.org/web/packages/GLDEX/GLDEX.pdf.*

| Model | $L_1$ | $L_2$ | $Pearson$ | $LeCam$ |
|---|---|---|---|---|
| Normal | 0.17476 | 0.047971 | 0.075765 | 0.034877 |
| Mix of Normals | 0.0958 | 0.025351 | 0.020596 | 0.011084 |
| Mix of Lambdas | 0.24636 | 0.063542 | 0.14132 | 0.054918 |

**Table 2.2.1** Values of several criteria for the models N, MixN and MixGLD
of the data from Table 1.1.

We see from this table that all four criteria are isotone in the sense that the MixN model fits best the data and the MixGLD fits worst. This can be explained by the fact that the criterion for selection of the mixed models was the best possible approximation of the first 4 moments of the e.d.f. and the MixGLD model is in this sense better than the MixN, as observed in Ning, Gao and Dudewicz (2008). The normal model approximates the first 2 moments only and this allows it to approximate the data source better in the sense of all four selected criteria. However, the relative differences between the accuracies of the models are evaluated by each criterion in a slightly different way. The relative difference between N and MixGLD is 20% through the prism of the $L_2$, while the Pearson criterion evaluates the same difference as 100% larger. No simple recipe can be given for the choice of the most adequate criterion. It usually requires a deep insight into the properties of the criteria and into the details of the studied application.

**Example 2.2.2**    Consider the e.d.f. $F_n$ defined by the data of Table 1.2. In addition to the normal model $N(\mu,\sigma^2)$ for these data given by (1.6), we consider the generalized lambda models $GLD(\theta)$ given by the percentile function

$$Q_\theta(y) = \theta_1 + \frac{y^{\theta_3} - (1-y)^{\theta_4}}{\theta_2}$$

for the parameters

$$\theta = (41.7897, 0.01134, 0.09853, 0.3606) \tag{2.27}$$

obtained on p. 97 of Karian and Dudewicz (2000). We use also the mixed generalized lambda model

$$MixGLD(\pi, \theta, \vartheta) = \pi GLD(\theta) + (1 - \pi)GLD(\vartheta)$$

with the component generalized lambda models given by the percentile functions

$$Q_\theta(y) = \theta_1 + \frac{y^{\theta_3} - (1-y)^{\theta_4}}{\theta_2} \quad \text{and} \quad Q_\vartheta(y) = \vartheta_1 + \frac{y^{\vartheta_3} - (1-y)^{\vartheta_4}}{\vartheta_2}.$$

for the parameters

$$\begin{aligned} \pi &= 0.4 \\ \theta &= (57.8233, 0.0076, 0.1432, 0.1356) \\ \vartheta &= (56.2136, 0.0193, 0.4601, 0.4838) \end{aligned} \tag{2.28}$$

introduced on p. 91 of Ning, Gao and Dudewicz (2008). The present models $F_0 \in \{N(\mu,\sigma^2), GLD(\theta), MixGLD(\pi, \theta, \vartheta)\}$ differ from those considered in the previous example but, similarly as above, they represent three different hypotheses about the sources of data from Table 1.2. Table 2.2.2 presents values of the distance or disparity $K(F, F_0)$, $AD(F, F_0)$ and $CM(F, F_0)$ given by (2.9), (2.20) and (2.22) for the d.f. $F(x) = F_n(x)$ from Figure 1.2 and the present models $F_0(x)$. Distribution functions of the models GLD and mixGLD were evaluated by the package GLDEX of the program R mentioned in Example 2.2.1.

| Model | $K(F_n, F_0)$ | $AD(F_n, F_0)$ | $CM(F_n, F_0)$ |
|---|---|---|---|
| Normal | 0.45177 | 0.38039 | 0.05513 |
| Lambda | 0.07917 | 0.39040 | 0.06034 |
| Mix of Lambdas | 0.21747 | 78.4530 | 0.78635 |

**Table 2.2.2** Values of several criteria for the models N, GLD and MixGLD
of the data from Table 1.2.

The preferences attributed to the models by the criteria are in Table 2.2.3 below. We see from there that the isotony similar to that observed in Table 2.2.1 takes place only for the Anderson-Darling and Cramér-von Mises criteria and that their preferences contradict those resulting from the Kolmogorov criterion. The extremely large value of the Anderson-Darling's $AD(F_n, F_0)$ in Table 2.2.2 underlines the importance of analysis of the criterion formulas and algorithms from the point of view of numerical stability, i.e. resistence (robustness) with respect to small computational inaccuracies. In the $AD(F_n, F_0)$ case, for $x \approx 1$ the empirical values $F_n(x)$ are close to 1 while the program computes the hypothetical values $F_0(x)$ by two orders closer to 1 without possibility to control the accuracy of computation. This obviously results into the uncontrolled increase of values during numerical integration in the formula

$$AD(F_n, F_0) = \int \frac{(F_n(x) - F_0(x))^2}{F_0(x)(1 - F_0(x))} \mathrm{d}x \quad \text{(c.f. (2.21))}$$

for $x$ such that the numerator is of the order of 1 while the denominator is of the order $< 10^{-2}$.

| Criterion | 1 | 2 | 3 |
|-----------|-----|-----|-----|
| $K(F_n, F_0)$ | Lambda | Mix of Lambdas | Normal |
| $AD(F_n, F_0)$ | Normal | Lambda | Mix of Lambdas |
| $CM(F_n, F_0)$ | Normal | Lambda | Mix of Lambdas |

**Table 2.2.3** The orders of preferences in Table 2.2.2.

Better performance of the simple lambda than of the mix of lambdas observed in Table 2.2.3 can be explained by the fact that the mix of lambdas is optimal in the sense of best approximation of the first 4 moments of the e.d.f. (c.f. Ning, Gao and Dudewicz (2008)). Similarly as mentioned above in the comments on Table 2.2.1, the best fit of the several first moments does not mean the best fit of d.f.'s which is measured by the goodness-of-fit criteria appearing in Tables 2.2.2 and 2.2.3. This is reflected also in the results of Section 8 where the fit of the models of this section and the data of Tables 1.1 and 1.2 is treated by the new criteria introduced in the next section.

## 3    Criteria of goodness-of-fit

This section studies more systematically those criteria of goodness-of-fit between the empirical evidence represented by the e.d.f. $F_n$ on the one hand and the hypothetical model specified by its d.f. $F_0$ on the other, that are measures of dissimilarity $D(F_n, F_0)$ between the distribution functions $F_n$ and $F_0$. Since $F_n \to F_0$ a.s. for $n \to \infty$ and, consequently, $D(F_n, F_0) \to 0$ stochastically as $n \to \infty$ for reasonable dissimilarity measures $D$, the statistics $T_n$ for testing the hypothesis $\mathcal{H}_0 : F_0$ on the basis of empirical evidence represented by $F_n$, are considered in the form $T_n = m_n D(F_n, F_0)$. Here $m_n \to \infty$ is an appropriate scaling sequence for which $T_n$ tends to a limit distribution. The value $Q(1 - \alpha)$ of the p.f. of this distribution is then used as a critical value of $T_n$ for the asymptotically $\alpha$-level test of hypothesis $\mathcal{H}_0$. Examples were given in Section 2, e.g. the Kolmogorov distance $K(F_n, F_0)$ and corresponding Kolmogorov-Smirnov statistic $T_n = \sqrt{n}K(F_n, F_0)$.

Similarly, if the empirical evidence is represented by a discrete distribution $\boldsymbol{p}_n$ and the hypothetical model is specified by the discrete distribution $\boldsymbol{p}_0$ then the dissimilarity between the former and the sequel is $D(\boldsymbol{p}_n, \boldsymbol{p}_0)$ and the statistic for testing the hypothesis $\mathcal{H}_0 : \boldsymbol{p}_0$ is $T_n = m_n D(\boldsymbol{p}_n, \boldsymbol{p}_0)$. Examples were given in Section 2, e.g. the Pearson distance $\chi^2(\boldsymbol{p}_n, \boldsymbol{p}_0)$ and the corresponding Pearson statistic $T_n = n\chi^2(\boldsymbol{p}_n, \boldsymbol{p}_0)$.

Goodness-of-fit criteria are not only studied for e.d.f.'s $F_n$ or related e.p.m.'s $\boldsymbol{p}_n$ simply representing the data sets $\{x_1, x_2, ..., x_n\}$ in a straightforward manner (1.2), but also for more sophisticated models $F$ and $\boldsymbol{p}$ obtained by data-based statistical inference like e.g. those obtained by maximum likelihood estimation. Therefore we deal in the rest of this section with arbitrary d.f.'s $F$ and p.m.'s $\boldsymbol{p}$.

## 3.1 Disparities, divergences and metric distances

To unify the treatment of the situation when goodness-of-fit is considered for d.f.'s $(F, F_0)$ or p.m.'s $(\boldsymbol{p}, \boldsymbol{p}_0)$, it is convenient to represent simultaneously both hypothetical models $F_0, \boldsymbol{p}_0$ by the corresponding general p.m.'s $P_0$, and alternative models $F, \boldsymbol{p}$ by the corresponding general p.m.'s $P$. Thus we deal in this subsection primarily with dissimilarity measures $D(P, P_0)$. We respect Convention 2.1 and use the notations (2.2) - (2.4) introduced there.

We define the class of **dissimilarities** of probability measures $P, P_0$ on $\mathcal{X}$ by

$$D_\phi(P, P_0) = \int_{\mathcal{X}} p_0 \phi\left(\frac{p}{p_0}\right) \mathrm{d}\lambda \quad \text{for } P \sim p, P_0 \sim p_0 \tag{3.1}$$

generated by continuous functions $\phi : (0, \infty) \mapsto [0, \infty)$ with continuous extension $0 \leq \phi(0) \leq \infty$, such that the integral (3.1) exists. In particular, for arbitrary d.f. $F$ and hypothetical d.f. $F_0$ on an interval observation space $\mathcal{X} = (a, b) \subseteq \mathbb{R}$,

$$D_\phi(F, F_0) = \int_a^b f_0(x) \phi\left(\frac{f(x)}{f_0(x)}\right) \mathrm{d}x \quad \text{for } F \sim f, F_0 \sim f_0, \tag{3.2}$$

and, in the discrete case, for

$$P \sim \boldsymbol{p} = (p_1, p_2, ..., p_k) \quad \text{and} \quad P_0 \sim \boldsymbol{p}_0 = (p_{01}, p_{02}, ..., p_{0k}) \tag{3.3}$$

c.f. (2.4),

$$D_\phi(P, P_0) = D_\phi(\boldsymbol{p}, \boldsymbol{p}_0) = \sum_{j=1}^k p_{0j} \phi\left(\frac{p_j}{p_{0j}}\right). \tag{3.4}$$

Let us clarify for which functions $\phi$ the dissimilarities (3.1) - (3.4) are disparities, divergences or distances in the sense of the previous section. Denote by $\boldsymbol{\Phi}$ the class of all differentiable functions $\phi : (0, \infty) \mapsto \mathbb{R}$ with continuous extension $\phi(0)$ and the property

$$(\phi'(t) - \phi'(1))\mathrm{sign}(t - 1) > 0 \quad \text{for all } t \in (0, \infty). \tag{3.5}$$

Then the standardized version

$$\tilde{\phi}(t) = \phi(t) - \phi(1) - \phi'(1)(t - 1) \tag{3.6}$$

of $\phi(t)$ is increasing in the domain $t \geq 0$ and decreasing in the domain $t \leq 0$, i.e. is positive for $t \in (0, \infty)$ except for $t = 1$ where $\tilde{\phi}(1) = 0$. Therefore the integral

$$D_{\tilde{\phi}}(P, P_0) = \int_{\mathcal{X}} p_0 \tilde{\phi}\left(\frac{p}{p_0}\right) \mathrm{d}\lambda = D_\phi(P, P_0) - \phi(1) \tag{3.7}$$

exists, takes on values in the closed interval $[0, \infty]$, and the dissimilarity $D_{\tilde{\phi}}(P, P_0)$ is reflexive in the sense that equality holds if and only if $P = P_0$. Hence the expressions $D_{\tilde{\phi}}(P, P_0)$ are well-defined disparities, and also the expressions $D_\phi(P, P_0)$ are disparities up to shifts $\phi(1)$. This justifies to speak about all dissimilarities $D_\phi(P, P_0), D_\phi(F, F_0)$ and $D_\phi(\boldsymbol{p}, \boldsymbol{p}_0)$ given by (3.1) - (3.4) for $\phi \in \boldsymbol{\Phi}$ as **disparities in the wide sense**, transformed by constant shifts $\phi(1)$ to disparities defined in the precise sense of the previous section. Of course, the subset

$$\boldsymbol{\Phi}_{\mathrm{disp}} = \{\phi \in \boldsymbol{\Phi} : \phi(1) = 0\}, \quad \boldsymbol{\Phi}_{\mathrm{disp}} \subset \boldsymbol{\Phi}$$

defines proper disparities by (3.1) - (3.4).

Let $\boldsymbol{\Phi}_{\mathrm{div}}$ be the class of differentiable convex functions $\phi : (0, \infty) \mapsto \mathbb{R}$ with continuous extension $\phi(0)$ and strict convexity at $t = 1$. Then $\phi(1) + \phi'(1)(t - 1)$ is the support straight line of the function $\phi(t)$ which is strictly smaller than $\phi(t)$ at all $t \neq 1$ due to the strict convexity of $\phi(t)$ at $t = 1$. However, the assumed convexity of $\phi(t)$ on the whole domain $(0, \infty)$ means that the function (3.6) is increasing on $(1, \infty)$ and decreasing on $(0, 1)$ so that (3.5) holds. Consequently,

$$\boldsymbol{\Phi}_{\mathrm{div}} \subset \boldsymbol{\Phi}_{\mathrm{disp}},$$

i.e. expression (3.7) is reflexive. As proved in Csiszár (1967) or Liese and Vajda (1987) (see also a new statistical proof in Vajda and Liese (2006)), the disparities (3.7) with $\phi \in \boldsymbol{\Phi}_{\mathrm{div}}$ satisfy the information processing law, i.e. they are *divergences* in the sense defined in the previous section.

**Example 3.1.1** The functions defined on $(0, \infty)$ by

$$\phi_0(t) = -\ln t \quad \text{and} \quad \phi_1(t) = t \ln t \tag{3.8}$$

with extensions $\phi_0(0) = \infty$ and $\phi_1(0) = 0$ satisfy condition (3.5). Thus they belong to $\boldsymbol{\Phi}$ and define the wide-sense disparities

$$D_1(P, P_0) := D_{\phi_1}(P, P_0) = \int_{\mathcal{X}} p \ln\left(\frac{p}{p_0}\right) d\lambda \tag{3.9}$$

and

$$D_0(P, P_0) := D_{\phi_0}(P, P_0) = \int_{\mathcal{X}} p_0 \ln\left(\frac{p_0}{p}\right) d\lambda = D_1(P_0, P). \tag{3.10}$$

Since $\phi_i(1) = 0$, both these functions belong to the subset $\boldsymbol{\Phi}_{\mathrm{disp}} \subset \boldsymbol{\Phi}$, so that (3.9) and (3.10) are proper disparities. Further, both functions $\phi_i(t)$ are differentiable and strictly convex on the domain $(0, \infty)$. Therefore they belong to $\boldsymbol{\Phi}_{\mathrm{div}}$ and (3.9) and (3.10) are divergences. In fact, $D_1(P, P_0)$ is known as the **information divergence** or **Kullback divergence**, and $D_0(P, P_0)$ is usually called the **reversed information divergence** or **reversed Kullback divergence**. Note that

$$T_n = n D_1(\boldsymbol{p}_n, \boldsymbol{p}_0) = \sum_{j=1}^{k} n p_{nj} \ln \frac{n p_{nj}}{n p_{0j}} = \sum_{j=1}^{k} \varphi_{nj} \ln \frac{\varphi_{nj}}{n p_{0j}} \tag{3.11}$$

is the well-known likelihood ratio test statistic for testing the hypothesis $\mathcal{H}_0 : P_0$ on the basis of observations represented by the e.p.m. $P_n$, where $\boldsymbol{p}_0$ and $\boldsymbol{p}_n$ are restrictions of the hypothetical and empirical p.m.'s $P_0$ and $P_n$ to the test cells $C_j = [c_j, c_{j+1})$, and $\varphi_{nj} = \#x_i \in C_j$ are the observed cell frequencies for $1 \leq j \leq k$.

**Example 3.1.2** The functions defined on $(0, \infty)$ by

$$\bar{\phi}_{-1}(t) = \frac{1}{t} \quad \text{and} \quad \bar{\phi}_2(t) = t^2$$

with extensions $\bar{\phi}_{-1}(0) = \infty$ and $\bar{\phi}_2(t) = 0$ satisfy condition (3.5). Thus they belong to $\boldsymbol{\Phi}$ and define the wide-sense disparities

$$\bar{D}_{-1}(P, P_0) := D_{\bar{\phi}_{-1}}(P, P_0) = \int_{\mathcal{X}} \frac{p_0^2}{p} d\lambda$$

and

$$\bar{D}_2(P, P_0) := D_{\bar{\phi}_2}(P, P_0) = \int_{\mathcal{X}} \frac{p^2}{p_0} d\lambda = \bar{D}_{-1}(P_0, P).$$

Since $\bar{\phi}_{-1}(1) = \bar{\phi}_2(1) = 1$, the functions $\delta_{-1}(t) := \bar{\phi}_{-1}(t) - 1$ and $\delta_2(t) = \bar{\phi}_2(t) - 1$ define the proper disparities

$$\Delta_{-1}(P, P_0) := D_{\delta_{-1}}(P, P_0) = \int_{\mathcal{X}} \frac{(p - p_0)^2}{p} d\lambda \tag{3.12}$$

and

$$\Delta_2(P, P_0) := D_{\delta_2}(P, P_0) = \int_{\mathcal{X}} \frac{(p - p_0)^2}{p_0} d\lambda. \tag{3.13}$$

The same proper disparities are defined by the standardized versions

$$\delta_{-1}^+(t) = \delta_{-1}(t) + t - 1 = \frac{(t-1)^2}{t} \quad \text{and} \quad \delta_2^+(t) = \delta_2(t) + 2(1 - t) = (t - 1)^2$$

belonging to $\boldsymbol{\Phi}_{\mathrm{disp}} \subset \boldsymbol{\Phi}$. Of course, the functions $\delta_{-1}(t)$ and $\delta_2(t)$ as well as $\delta_{-1}^+(t)$ and $\delta_2^-(t)$ belong to $\boldsymbol{\Phi}_{\mathrm{div}}$ and define the Pearson divergences of Section 2.1, part $\mathbf{d}$ (briefly 2.1.($\mathbf{d}$)),

$$\Delta_2(P, P_0) = \chi^2(P, P_0) \ (\text{cf. (2.14)}) \quad \text{and} \quad \Delta_{-1}(P, P_0) = R\chi^2(P, P_0) \ (\text{cf. (2.17)}).$$

**Example 3.1.3 Power divergences**   The functions defined on $(0, \infty)$ by

$$\phi_\alpha(t) = \frac{t^\alpha - 1}{\alpha(\alpha - 1)} \quad \text{for the powers } \alpha \in \mathbb{R} - \{0, 1\} \tag{3.14}$$

and for the remaining powers by

$$\phi_1(t) = t \ln t \quad \text{and} \quad \phi_0(t) = -\ln t \quad \text{(cf. Example 3.1.1))} \tag{3.15}$$

belong to $\Phi$ defined in Example 3.1.1. They satisfy the condition $\phi_\alpha(1) = 0$ so that they belong also to the subset $\Phi_{\text{disp}} \subset \Phi$. Since they are differentiable and strictly convex on $(0, \infty)$, they belong in fact also to $\Phi_{\text{div}}$ and define the divergences

$$D_\alpha(P, P_0) := D_{\phi_\alpha}(P, P_0) = \frac{\int_{\mathcal{X}} p^\alpha p_0^{1-\alpha} \mathrm{d}\lambda - 1}{\alpha(\alpha - 1)} \quad \text{for } \alpha \in \mathbb{R} - \{0, 1\} \tag{3.16}$$

and

$$D_1(P, P_0) := D_{\phi_1}(P, P_0) = \int_{\mathcal{X}} p \ln \left( \frac{p}{p_0} \right) \mathrm{d}\lambda = D_0(P_0, P) := D_{\phi_0}(P_0, P) \quad \text{for } \alpha \in \{0, 1\}. \tag{3.17}$$

The classes of modified power divergences $\alpha D_\alpha(P, P_0), \alpha > 0$ are one-to-one related to the logarithmic power divergences

$$R_\alpha(P, P_0) = \frac{\ln[\alpha(\alpha - 1)D_\alpha(P, P_0) + 1]}{\alpha - 1} = \frac{\ln \int_{\mathcal{X}} p^\alpha p_0^{1-\alpha} \mathrm{d}\lambda}{\alpha - 1}, \quad \alpha > 0 \tag{3.18}$$

of Rényi (1961), where $R_1(P, P_0) = \lim_{\alpha \uparrow 1} R_\alpha(P, P_0) = D_1(P, P_0)$. The slightly restricted versions $\alpha D_\alpha(P, P_0), \alpha > 0$, were used in the 1960-ies and 1970-ies by several authors, e.g. by Perez (1967). The essential parts of the power divergences

$$H_\alpha(P, P_0) = \alpha(\alpha - 1)D_\alpha(P, P_0) + 1 = \int_{\mathcal{X}} p^\alpha p_0^{1-\alpha} \mathrm{d}\lambda, \quad \alpha \in \mathbb{R} \tag{3.19}$$

called **Hellinger integrals** were used e.g. by Chernoff (1952), Vajda (1971) and Liese (1982). The Hellinger integrals are skew symmetric about $\alpha = 1/2$ in the sense that

$$H_\alpha(P, P_0) = H_{1-\alpha}(P_0, P), \quad \alpha \in \mathbb{R}. \tag{3.20}$$

The power divergences (3.16), (3.17) are skew symmetric in the same sense, i.e.

$$D_\alpha(P, P_0) = D_{1-\alpha}(P_0, P), \quad \alpha \in \mathbb{R}. \tag{3.21}$$

The skew-symmetrization of the formerly used divergences $\alpha D_\alpha(P, P_0), \alpha > 0$, resulting into the class of power divergences (3.16),(3.17) was introduced by Cressie and Read (1984), and the power divergences are in the last two decades used as standard representatives of the whole class of divergences $D_\phi(P, P_0), \phi \in \Phi_{\text{div}}$. A similar skew-symmetrization of the Rényi divergences (3.18) resulting into the formula

$$R_\alpha(P, P_0) = \frac{\ln \int_{\mathcal{X}} p^\alpha p_0^{1-\alpha} \mathrm{d}\lambda}{\alpha(\alpha - 1)}, \quad \alpha \in \mathbb{R} \tag{3.22}$$

was introduced by Liese and Vajda (1987) who used them among other as auxiliary tools for rigorous proofs of a number of general properties of the power divergences (3.21).

**Remarks**   **(i)** Since the functions (3.15) are the same as in Example 3.1.1, the members $D_1(P, P_0)$ and $D_0(P, P_0)$ of the power divergence family are the **Kullback** and **reversed Kullback divergences** introduced in Example 3.1.1.

**(ii)** Further,

$$D_\alpha(P, P_0) = \Delta_\alpha(P, P_0)/2 \quad \text{for } \alpha = -1 \text{ and } \alpha = 2$$

where on the right are the disparities introduced in Example 3.1.2. Therefore $2D_2(P, P_0)$ and $2D_{-1}(P, P_0)$ are the Pearson and reversed Pearson divergences of 2.1(d)).

(iii) Another well-known member of the family of power divergences not mentioned before is the **Hellinger divergence**

$$D_{1/2}(P, P_0) = 4H(P, P_0) = 4\int_{\mathcal{X}} \left(\sqrt{p} - \sqrt{p_0}\right)^2 d\lambda, \tag{3.23}$$

which in the discrete case (2.4) reduces to

$$D_{1/2}(\boldsymbol{p}, \boldsymbol{p}_0) = 4 = 4\sum_{j=1}^{k} \left(\sqrt{p_j} - \sqrt{p_{0j}}\right)^2. \tag{3.24}$$

The divergence $H(\boldsymbol{p}, \boldsymbol{p}_0)$ was introduced by Matusita (1956) but it is better known as **squared Hellinger distance**. Indeed,

$$\sqrt{H(P, P_0)} = ||\sqrt{p} - \sqrt{p_0}||_2$$

is the $L_2$-distance of the square roots of p.d.f.'s and, as such, is a metric distance. Of course, $\sqrt{D_{1/2}(\boldsymbol{p}, \boldsymbol{p}_0)}$ is a metric distance too.

(iv) The expression

$$T_n = nD_{1/2}(\boldsymbol{p}_n, \boldsymbol{p}_0)/2 = 2n\sum_{j=1}^{k} \left(\sqrt{p_j} - \sqrt{p_{0j}}\right)^2 \tag{3.25}$$

is known as the **Freeman-Tukey statistic** for testing the hypothesis $\mathcal{H}_0 : P_0$ on the basis of observations represented by the e.p.m. $P_n$, where $\boldsymbol{p}_0$ and $\boldsymbol{p}_n$ are the restrictions of the hypothetical and empirical p.m.'s $P_0$ and $P_n$ on the test cells $C_j = [c_j, c_{j+1})$ for $1 \le j \le k$.

(v) The square root $\sqrt{D_{1/2}(\boldsymbol{p}, \boldsymbol{p}_0)}$ is the only member among all powers of all power divergences which is a distance in the sense introduced above (cf Definition 2.1). The unicity follows already from the fact that $D_{1/2}(P, P_0)$ is the only power divergence which is symmetric in the variables $P, P_0$.

(vi) The nonnegative standardized versions of the functions (3.14) are

$$\tilde{\phi}_\alpha(t) = \frac{t^\alpha - \alpha(t-1) - 1}{\alpha(\alpha-1)} \quad \text{for } \alpha \in \mathbb{R} - \{0, 1\} \tag{3.26}$$

and those corresponding to (3.15) are the limits

$$\tilde{\phi}_1(t) = t\ln t - t + 1 \quad \text{and} \quad \tilde{\phi}_0(t) = -\ln t + t - 1 \tag{3.27}$$

of $\tilde{\phi}_\alpha(t)$ for $\alpha \to 0$ and $\alpha \to 1$. Obviously

$$D_{\tilde{\phi}_\alpha}(P, P_0) = D_{\phi_\alpha}(P, P_0) = D_\alpha(P, P_0) \quad \text{for all } \alpha \in \mathbb{R}. \tag{3.28}$$

## 3.2 Metricity and robustness

The power divergences $D_\alpha(P, P_0), \alpha \in \mathbb{R}$, do not represent all aspects of the class of all divergences $D_\phi(P, P_0), \phi \in \boldsymbol{\Phi}_{\text{div}}$. For example, $\sqrt{D_{1/2}(P, P_0)}$ is the only power divergence which is a metric distance. This might suggest that the metricity of a divergence is a rare property. But, in fact, the class of all divergences contains uncountably many of them with this property. For example, all functions

$$\varphi_\beta(t) = \frac{\text{sign } \beta}{1-\beta} \left[(t^{1/\beta} + 1)^\beta - 2^{\beta-1}(t+1)\right] \quad \text{for } -\infty < \beta \le 2 \tag{3.29}$$

with the terms for $\beta = 0$ and $\beta = 1$ obtained by the continuous extension rule as

$$\varphi_0(t) = |t - 1|/2 \quad \text{and} \quad \varphi_1(t) = t\ln t + (t+1)\ln\frac{2}{t+1} \tag{3.30}$$

belong to $\boldsymbol{\Phi}_{\mathrm{div}}$. The square roots $\sqrt{D_{\phi_\beta}(P, P_0)}$ of the corresponding divergences are metric distances (see Vajda (2009)). LeCam's divergence of 2.1.**(g)** is among them, since

$$D_{\varphi_{-1}}(P, P_0) = LC(P, P_0).$$

The total variation function $\varphi_0(t)$ formally does not belong to $\boldsymbol{\Phi}_{\mathrm{div}}$ because it is not differentiable at $t = 1$, but this may be cured by defining this derivative as the mean of the left- and right-hand derivatives,

$$\varphi_0'(1) = \frac{\varphi_0'(1+) + \varphi_0'(1-)}{2} = \frac{1}{2} - \frac{1}{2} = 0. \tag{3.31}$$

Let us now look at the robustness of the testing or estimation based on the disparity statistics

$$T_{\phi,n} = nD_\phi(P_n, P_0), \quad \phi \in \boldsymbol{\Phi}_{\mathrm{disp}}$$

reflecting the proximity of the hypothetical p.m. $P_0$ and e.p.m. $P_n$, or their special forms

$$T_{\phi,n} = nD_\phi(\boldsymbol{p}_n, \boldsymbol{p}_0) = \sum_{j=1}^k np_{0j}\phi\left(\frac{np_{nj}}{np_{0j}}\right) = \sum_{j=1}^k np_{0j}\phi\left(\frac{\varphi_{nj}}{np_{0j}}\right), \quad \phi \in \boldsymbol{\Phi}_{\mathrm{disp}}$$

using restrictions $\boldsymbol{p}_0$ and $\boldsymbol{p}_n$ of $P_0$ and $P_n$ on the test cells $C_j = [c_j, c_{j+1})$ where $\varphi_{nj} = \#x_i \in C_j$ are the observed cell frequencies for $1 \le j \le k$.

If $\phi \in \boldsymbol{\Phi}_{\mathrm{div}}$, i.e. if $D_\phi(P, P_0)$ is a divergence then $\phi(t)$ is always unbounded and its derivative $\phi'(t)$ is usually unbounded on the domain $(0, \infty)$. For example, all power divergence functions $\phi_\alpha(t)$ as well as their derivatives $\phi_\alpha'(t)$ are unbounded. The statistical disparity measures were introduced by Lindsay (1994) and more systematically investigated by Menéndes et al. (1998). In these papers it is argued that the robustness of statistical inference based on minimization of disparities between models requires bounded generating functions $\phi \in \boldsymbol{\Phi}_{\mathrm{disp}}$, or at least bounded derivatives $\phi\prime$, because these two functions represent the criterion function and the influence function of robust statistics. Thus, from the point of robustness of statistical decisions based on disparity statistics, the attention is concentrated on the functions $\rho(t-1) \in \boldsymbol{\Phi}_{\mathrm{disp}}$ for the classical criterial $\rho$-functions of robust statistics leading to bounded influence functions $\psi(t)$ proportional to the derivatives $\rho'(t)$.

**Example 3.2.1** A classical example is the family of **Huber $\rho$-functions**

$$\rho_k(t) = \boldsymbol{I}(|t| \le k)t^2, \quad k > 0$$

smoothly extended as linear functions in the domain $\{t \in \mathbb{R} : |t| > k\}$ with the constantly bounded influences $\psi_k(t) = \rho_k'(t) = 2k$ of the observations $t$ from this domain (see Hampel et al. (1986) or Jurečková and Sen (1996)). The Huber functions define the family

$$\phi_k(t) = \rho_k(t-1), \quad k > 0 \tag{3.32}$$

of functions from $\boldsymbol{\Phi}_{\mathrm{div}}$ with the bounded derivatives

$$\psi_k(t) = \phi_k'(t) = 2[(t-1)\boldsymbol{I}(|t-1| \le k) + k\boldsymbol{I}(|t-1| > k), \quad k > 0. \tag{3.33}$$

These functions generate the family of robust divergences $D_{\phi_k}(P, P_0), k > 0$.

**Example 3.2.2** Another classical example is the family of $\rho$-functions

$$\rho_\alpha(t) = (\alpha-1)t\boldsymbol{I}(t < 0) + \alpha t\boldsymbol{I}(t > 0), \quad 0 < \alpha < 1$$

leading to the robust statistical inference based on the so-called **regression quantiles** (see Jurečková and Sen (1996)). The total variation generating function is the special case

$$\varphi_0(t) = \rho_{1/2}(t-1).$$

We put

$$\phi_\alpha(t) = \rho_\alpha(t-1), \quad 0 < \alpha < 1 \tag{3.34a}$$

and similarly as in (3.31), we use the generalized form

$$\phi'_\alpha(1) = \frac{\phi'_\alpha(1+) + \phi'_\alpha(1-)}{2} = \alpha - \frac{1}{2} \tag{3.35}$$

to extend the derivative $\phi'_\alpha(t)$ of the family of robust divergence generating functions to $t = 1$ in order to achieve the formal validity for including this family in $\mathbf{\Phi}_{\mathrm{div}}$. The derivatives

$$\psi_\alpha(t) = \phi'_\alpha(t) = (\alpha - 1)\boldsymbol{I}(t < 1) + \alpha\boldsymbol{I}(t > 1), \quad 0 < \alpha < 1 \tag{3.36}$$

of these functions are bounded on the domain $(0, \infty)$ so that the functions (3.34a) generate the family of robust divergences $D_{\phi_\alpha}(P, P_0)$, $0 < \alpha < 1$.

# 4    Disparities based on partitions

In the previous section we assessed goodness-of-fit between two statistical models given by p.m.'s $P$ and $P_0$ by the disparity, divergence or distance

$$D_\phi(P, P_0) = \int_{\mathcal{X}} p_0\phi\left(\frac{p}{p_0}\right)\mathrm{d}\lambda \tag{4.1}$$

where the concrete type depends on the extended real valued function $\phi \in \mathbf{\Phi}$ and, according to Convention 2.1, $p, p_0$ are densities of $P, P_0$ given by (2.2) and $p_0$ is positive on $\mathcal{X}$. Definition (2.2) of the densities $p, p_0$ means that for every $A \subset \mathcal{X}$

$$P(A) = \int_A p\mathrm{d}\lambda \ \text{ and } \ P_0(A) = \int_A p_0\mathrm{d}\lambda \ \text{ for } \ \lambda = P + P_0. \tag{4.2}$$

In some situations it is necessary to restrict the p.m.'s $P$ and $P_0$ on a partition $\mathcal{P} = \{C_1, ..., C_k\}$ of $\mathcal{X}$ into disjoint cells $C_j$ , resulting into the quantizations

$$\boldsymbol{p} = (p_j := P(C_j) : 1 \le j \le k) \ \text{ and } \ \boldsymbol{p}_0 = (p_{0j} := P_0(C_j) : 1 \le j \le k) \tag{4.3}$$

of these p.m.'s and to the reduced disparity, divergence or distance

$$D_\phi(\boldsymbol{p}, \boldsymbol{p}_0) = \sum_{j=1}^{k} p_{0j}\phi\left(\frac{p_j}{p_{0j}}\right). \tag{4.4}$$

Partitioning of the observation space means that observations $x \in \mathcal{X}$ are replaced by the indices of the partition sets containing these observations, i.e. by

$$T(x) \in \{1, 2, ..., k\} \quad \text{where} \quad T^{-1}(j) = C_j \quad \text{for } \ 1 \le j \le k. \tag{4.5}$$

Among other this means that if $\phi$ generates the divergence $D_\phi(P, P_0)$ , then the information processing law implies that

$$D_\phi(\boldsymbol{p}, \boldsymbol{p}_0) \le D_\phi(P, P_0) \tag{4.6}$$

where the equality holds if and only if the partition $\mathcal{P}$ is statistically sufficient for $(P, P_0)$.

The situation described above takes place when the first of the p.m.'s is an e.p.m., i.e. when it is uniform on the observation support set $S_n = \{x_1, x_2, ..., x_n\}$ according to (1.1). Then (4.2) holds for

$$p(x) = p_n(x) := \boldsymbol{I}(x \in S_n) \ \text{ and } \ p_0(x) := \boldsymbol{I}(x \notin S_n)$$

because
$$(P_n + P_0)(S_n \cap A) = P_n(A) \text{ and } (P_n + P_0)((\mathcal{X} - S_n) \cap A) = P_n(A) \text{ for every } A \subset \mathcal{X}.$$

This, together with the fact that $P_n$ is supported by $S_n$ and $P_0$ is supported by $\mathcal{X} - S_n$ implies that

$$
\begin{aligned}
D_\phi(P_n, P_0) &= \int_{\mathcal{X}} p_0 \, \phi\left(\frac{p_n}{p_0}\right) \mathrm{d}(P_n + P_0) \\
&= \int_{S_n} \mathbf{I}(x \notin S_n) \, \phi\left(\frac{\mathbf{I}(x \in S_n)}{\mathbf{I}(x \notin S_n)}\right) \mathrm{d}P_n \\
&+ \int_{\mathcal{X} - S_n} \mathbf{I}(x \notin S_n) \, \phi\left(\frac{\mathbf{I}(x \in S_n)}{\mathbf{I}(x \notin S_n)}\right) \mathrm{d}P_0 \\
&= \int_{S_n} 0 \, \phi\left(\frac{1}{0}\right) \mathrm{d}P_n + \int_{\mathcal{X} - S_n} 1 \, \phi\left(\frac{0}{1}\right) \mathrm{d}P_0 \\
&= \lim_{t \to \infty} \frac{\phi(t)}{t} + \phi(0),
\end{aligned}
\tag{4.7}
$$

where in the last line we replaced the undefined expression by the limit on the basis of the principle of continuous extension. The existence of the limit is guaranteed only for convex $\phi$, i.e. for divergences, but even then the value (4.7) is constant, often infinite (for the power divergences it is finite only for the powers $0 < \alpha < 1$). Thus, without any further specification or restriction, the disparities, divergences or distances $D_\phi(P_n, P_0)$ are meaningless for statistical inference. Hence for the rest of the chapter we adopt the following convention.

**Convention 4.1** The disparities, divergences or distances $D_\phi(P_n, P_0)$ are replaced by their discrete versions (4.4) resulting from finite partitions $\mathcal{P} = \{C_1, ..., C_k\}$ of the observation space $\mathcal{X}$ by the quantization rule (4.3). Further, we restrict ourselves to **real valued observations** and assume interval observation spaces $\mathcal{X} = (a, b) \subseteq \mathbb{R}$ on which statistical models $P_n, P, P_0$ are given by d.f.'s $F_n, F, F_0$ where both $F$ and $F_0$ are assumed to have positive densities $f > 0$ and $f_0 > 0$. Moreover, we restrict ourselves to the interval partitions

$$\mathcal{P} = \{C_j := (c_{j-1}, c_j] : 1 \leq j \leq k\} \text{ for } a = c_0 < c_1 < ... < c_{k-1} < c_k = b \tag{4.8}$$

where the quantization rule (4.3) takes on the form

$$
\begin{aligned}
\boldsymbol{p} &= (p_j := F(c_j) - F(c_{j-1}) : 1 \leq j \leq k), &\tag{4.9} \\
\boldsymbol{p}_n &= (p_{nj} := F_n(c_j) - F_n(c_{j-1}) : 1 \leq j \leq k) &\tag{4.10} \\
\boldsymbol{p}_0 &= (p_{0j} := F_0(c_j) - F_0(c_{j-1}) : 1 \leq j \leq k). &\tag{4.11}
\end{aligned}
$$

By (4.4), the **disparities, divergences** or **distances** $D_\phi(\boldsymbol{p}_n, \boldsymbol{p}_0)$ depend on the frequencies of the observations $x_1, x_2, ..., x_n$ in the intervals of the partition $\mathcal{P}$ but not on the order of these observations. Therefore the vector $(x_1, x_2, ..., x_n)$ of observations can be replaced by the order statistics $(x_{n:1}, x_{n:2}, ..., x_{n:n})$. Further, the **hypothetical model** $F_0$, which is compared with the **alternative model** $F$ or the **empirical model** $F_n$, is known and by the assumptions the function $F_0(y)$ is strictly increasing on the observation space $\mathcal{X} = (a, b)$. Hence it can be used to transform in a one-to-one manner this observation space into the simple standardized observation space $\mathcal{Y} = (F_0(a), F_0(b)) = (0, 1)$ commonly used in the literature dealing with testing hypotheses $\mathcal{H}_0 : F_0$ against alternatives $\mathcal{A} : F$. The d.f. $F$ governs in an i.i.d. manner the random observations $X_1, X_2, ..., X_n$ generating the realizations $x_1, x_2, ..., x_n$. In this new observation space we deal with the ordered observations

$$F_0(a) := 0 = Y_0 < Y_1 := F_0(X_{n:1}) \leq ... \leq Y_n := F_0(X_{n:n}) < 1 = Y_{n+1} := F_0(b) \tag{4.12}$$

and with the hypotheses $\mathcal{H}_0 : F_0(Q_0)$, alternatives $\mathcal{A} : F(Q_0)$ and e.d.f.'s $F_n(Q_0)$, all defined on $[0, 1]$ by means of the increasing **hypothetical percentile function** (briefly, **h.p.f.**)

$$Q_0(y) = F_0^{-1}(y) \text{ on } [0, 1]. \tag{4.13}$$

This motivates the next convention which will also hold for the remainder of this chapter.

**Convention 4.2**  We assume without loss of generality that we test the hypothesis of uniformity $\mathcal{H}_0 : F_0$ with constant p.d.f. $f_0(y) = 1$ and linear h.p.f. $Q_0(y) = F_0(y) = y$ on $(0,1)$ against the alternative $\mathcal{A} : F$ with a p.d.f. $f(y)$ positive on $(0,1)$ and the percentile function $Q(y) = F^{-1}(y)$ increasing on $[0,1]$. The testing is based on the e.d.f. $F_n(y)$ on $[0,1]$ with jumps $1/n$ at the points

$$0 < Y_1 \leq ... \leq Y_n < 1 \tag{4.14}$$

obtained by ordering the random observations

$$X_1, X_2, ..., X_n \text{ i.i.d. by the p.d.f. } f(y) \text{ on the observation space } (0,1). \tag{4.15}$$

Alternatively, the testing can be based on the one-to-one related **empirical percentile function** (briefly, **e.p.f.**)

$$Q_n(y) = F_n^{-1}(y) = \inf\{z \in (0,1] : F_n(z) \geq y\} \text{ on } [0,1]. \tag{4.16}$$

Consequently

$$Q_n\left(\frac{j}{n}\right) = Y_j, \quad 0 \leq j \leq n \tag{4.17}$$

and the disparities, divergences or distances $D_\phi(P, P_0)$ and $D_\phi(P_0, P)$ are given by the formulas

$$D_\phi(P, P_0) = \int_0^1 \phi(p)\,\mathrm{d}y \quad \text{and} \quad D_\phi(P_0, P) = \int_0^1 p\phi\left(\frac{1}{p}\right)\mathrm{d}y. \tag{4.18}$$

*This is the the basic conceptual framework for the rest of the chapter.*

The information available to the statistician when he faces the problem of testing hypothesis $\mathcal{H}_0 : F_0$ is represented by the d.f.'s $F_n$ and $F_0$. It is used by him to calculate the decision tool $D_\phi(\boldsymbol{p}_n, \boldsymbol{p}_0)$ or $T_n = m_n D_\phi(\boldsymbol{p}_n, \boldsymbol{p}_0)$ using the rules (4.4), (4.10) and (4.11). Naturally, this procedure is simplified if either the distribution $\boldsymbol{p}_0$ or $\boldsymbol{p}_n$ is uniform,

$$\boldsymbol{p}_0 = \left(\frac{1}{k}, \frac{1}{k}, ..., \frac{1}{k}\right) \quad \text{or} \quad \boldsymbol{p}_n = \left(\frac{1}{k}, \frac{1}{k}, ..., \frac{1}{k}\right) \quad \text{for some } k > 1 \tag{4.19}$$

where $k$ may increase with the sample size $n$, i.e. the dependence

$$k = k_n \tag{4.20}$$

is admitted but the subscript $n$ is suppressed unless it plays an explicit role. These two possibilities are mutually exclusive for large $k$ and are studied separately in the next two subsections.

## 4.1  Partitioning by hypothetical percentiles

Let us start with the first case considered in (4.19). It takes place if the partition (4.8) is defined by the $k+1$ cutpoints

$$c_j = Q_0\left(\frac{j}{k}\right) = \frac{j}{k}, \quad 0 \leq j \leq k \quad \text{for the h.p.f. } Q_0 \text{ (c.f. (4.13))}, \tag{4.21}$$

which are the **hypothetical percentiles** of the uniformly distributed orders $0, 1/k, ..., (k-1)/k, 1$. Thus, the partitioning by uniformly distributed hypothetical percentiles leads to the discrete alternative, empirical and hypothetical distributions

$$\boldsymbol{p} = \left(p_j := F\left(\frac{j}{k}\right) - F\left(\frac{j-1}{k}\right) : 1 \leq j \leq k\right), \tag{4.22}$$

$$\boldsymbol{p}_n = \left(p_{nj} := F_n\left(\frac{j}{k}\right) - F_n\left(\frac{j-1}{k}\right) : 1 \leq j \leq k\right), \tag{4.23}$$

$$\boldsymbol{p}_0 = \left(p_{0j} := F_0\left(\frac{j}{k}\right) - F_0\left(\frac{j-1}{k}\right) = \frac{1}{k} : 1 \leq j \leq k\right) \tag{4.24}$$

respectively. Based on formula (4.4), the disparities, divergences or distances $D_\phi(\boldsymbol{p}_n, \boldsymbol{p}_0)$ and $D_\phi(\boldsymbol{p}_0, \boldsymbol{p}_n)$ are given by the formulas

$$D_\phi(\boldsymbol{p}_n, \boldsymbol{p}_0) = k \sum_{j=1}^{k} \phi\left(kp_{nj}\right) \quad \text{(cf. (4.23), (4.24))} \tag{4.25}$$

and

$$D_\phi(\boldsymbol{p}_0, \boldsymbol{p}_n) = \sum_{j=1}^{k} p_{nj}\phi\left(\frac{1}{kp_{nj}}\right) \quad \text{(cf. (4.23), (4.24))} \tag{4.26}$$

where the latter is meaningful only if all $p_{nj}$ are positive.

The probability that all $p_{nj}$ are positive decreases when $k$ increases and vanishes for $k \geq n+1$, since there are only $n$ observations, i.e. $F_n$ has at most $n$ jumps. Therefore the version (4.26) is not used in the sequel and attention is restricted to the goodness-of-fit criteria (4.25) or their scaled versions $T_{\phi,n} = nD_\phi(\boldsymbol{p}_n, \boldsymbol{p}_0)$. The above introduced concrete statistics (2.16), (2.21), (2.23), (3.11) and (3.25) are examples of such scaled versions if their cells $C_j$ are specified by the cutpoints (4.21).

## 4.2 Partitioning by empirical percentiles.

In this subsection we study the second possibility considered in (4.19). It takes place if the partition (4.8) is defined by the same number of $k+1$ cutpoints

$$0 = c_0 < c_1 < ... < c_{k-1} < c_k = 1 \tag{4.27}$$

as in Subsection 4.1, but the first $k$ of them are the **empirical percentiles** of the uniformly distributed orders $0, 1/k, ..., (k-1)/k$, i.e.,

$$c_j = \begin{cases} Q_n\left(\frac{j}{k}\right), & 0 \leq j \leq k-1 \quad \text{for the e.p.f. } Q_n \text{ (c.f. (4.16))} \\ \\ 1 \quad \text{for } j = k \end{cases} \tag{4.28}$$

This formula is not a complete parallel to (4.21) of the previous subsection, because here the last cutpoint is $c_k = 1$ and not $Q_n(k/k) = Q_n(1) = Y_n < 1$. (See in this respect Remark 4.2.1 below.) The cutpoints formula simplifies when $k$ divides $n$. Unless otherwise explicitly stated, we assume that

$$k = k_n = \frac{n}{m} \quad \text{for } m = m_n = 1, 2, ... \quad \text{(cf. (4.20))}. \tag{4.29}$$

In accordance with the agreement above, the subscript $n$ is suppressed in $k_n, m_n$ and all expressions involving $k_n, m_n$ unless it is explicitly needed to display.

By the definitions of $F_n, F_0$, under the assumption (4.29) the cutpoints formula (4.28) implies

$$c_j = Q_n\left(\frac{jm}{n}\right) = Y_{jm}, \quad 0 \leq j \leq k-1 \tag{4.30}$$

so that

$$F_0(c_j) = c_j = Y_{jm}, \quad 0 \leq j \leq k-1. \tag{4.31}$$

Since $c_k = 1$ so that $F_n(c_k) = 1 = km/n$, we get from (4.10) and (4.30)

$$\boldsymbol{p}_n = \left(p_{nj} := \frac{jm}{n} - \frac{(j-1)m}{n} = \frac{m}{n} = \frac{1}{k} : 1 \leq j \leq k\right). \tag{4.32}$$

Similarly, from (4.11) and (4.31) we get

$$\boldsymbol{p}_0 = \left(p_{0j} := Y_{jm} - Y_{(j-1)m} : 1 \leq j \leq k-1 \ \text{ and } \ p_{0k} := 1 - Y_{(k-1)m}\right). \tag{4.33}$$

In this formula $p_{0k} := 1 - Y_{(k-1)m}$ cannot be replaced by $p_{0k} := Y_{km} - Y_{(k-1)m}$ because $c_k = 1 = F_0(c_k)$ is almost surely greater than $Y_{km} = Y_n$. Notice that under the assumption (4.29) the cutpoints formula (4.28) implies that each cell of the partition of the interval contains exactly $m$ observations.

By (4.4), the disparities, divergences or distances $D_\phi(\boldsymbol{p}_n, \boldsymbol{p}_0)$ and $D_\phi(\boldsymbol{p}_0, \boldsymbol{p}_n)$ of the distributions (4.32) and (4.33) are given by the formulas

$$D_\phi(\boldsymbol{p}_n, \boldsymbol{p}_0) = D_{\phi^*}(\boldsymbol{p}_n, \boldsymbol{p}_0) \quad \text{for} \quad \phi^*(t) = t\phi\left(\frac{1}{t}\right) \quad \text{on } (0, \infty) \tag{4.34}$$

where

$$\begin{aligned}
D_\phi(\boldsymbol{p}_0, \boldsymbol{p}_n) &= \sum_{j=1}^{k-1} \frac{1}{k} \phi\left(k(Y_{jm} - Y_{(j-1)m})\right) + \frac{1}{k}\phi\left(k(1 - Y_{(k-1)m})\right) \\
&= \frac{m}{n} \sum_{j=1}^{k} \phi\left(\frac{n}{m}(Y_{jm} - Y_{(j-1)m})\right) + \frac{m}{n}\phi\left(\frac{n}{m}(1 - Y_{(j-1)m})\right).
\end{aligned} \tag{4.35}$$

The version (4.34) leads to a too complicated formula in terms of the original function $\phi$. Thus for theoretical analysis as well as for practical applications it is more convenient to work with version (4.35). The stochastic differences $Y_{jm} - Y_{(j-1)m}$ are generally referred to as the $m$-spacings.

The disparities, divergences or distances (4.35) define the $m$-spacings based goodness-of-fit statistics

$$\begin{aligned}
T_\phi^{(m)} &= T_{\phi,n}^{(m)} = nD_\phi(\boldsymbol{p}_0, \boldsymbol{p}_n) \\
&= m\sum_{j=1}^{k-1} \phi\left(\frac{n}{m}(Y_{jm} - Y_{(j-1)m})\right) + m\phi\left(\frac{n}{m}(1 - Y_{(k-1)m})\right).
\end{aligned} \tag{4.36}$$

**Remark 4.2.1** If in full analogy with (4.21) the cutpoint scheme (4.27, (4.28) is replaced by

$$c_j = Q_n\left(\frac{j}{k}\right), \quad 0 \le j \le k \quad \text{for the e.p.f. } Q_n \text{ (c.f. (4.16)}$$

then $c_k = Y_n < 1$, so that the components $p_{0j} = F_0(c_j) - F_0(c_{j-1}) = c_j - c_{j-1}$ of the hypothetical distribution $\boldsymbol{p}_0$ satisfy the strict inequality

$$\sum_{j=1}^{k} p_{0j} = c_k = Y_n < 1$$

i.e. they cannot be normalized to 1. This can be solved by adding the cutpoint $c_{k+1} = 1$ when the collection of cutpoints

$$0 = c_0 < c_1 < \dots < c_{k-1} < c_k < c_{k+1} = 1$$

generates hypothetical and empirical distributions $\boldsymbol{p}_0$ and $\boldsymbol{p}_n$ with $k+1$ components, both normalized to 1. However, then the last component of the empirical distribution is

$$p_{n,k+1} = F_n(c_{k+1}) - F_n(c_k) = F_n(1) - F_n(Y_n) = 0$$

so that the empirical distribution is not uniform and its components are not strictly positive. Nevertheless, for $\phi \in \Phi_{\text{div}}$ the $\phi$-divergences $D_\phi(\boldsymbol{p}_0, \boldsymbol{p}_n)$ remain to be well defined by the formula

$$D_\phi(\boldsymbol{p}_0, \boldsymbol{p}_n) = D_{\phi^*}(\boldsymbol{p}_n, \boldsymbol{p}_0) = \sum_{j=1}^{k} \frac{1}{k} \phi\left(k(Y_{jm} - Y_{(j-1)m})\right) + (1 - Y_n)\phi^*(0) \tag{4.37}$$

for the function $\phi^*(t)$ defined in (4.34) (for details see e.g. Liese and Vajda (2006)). Thus the $\phi$-divergences with finite limit $\phi^*(0) = \lim_{t \to \infty} \phi(t)/t$ define meaningful spacings-based divergence statistics

$$T_\phi^{*(m)} = T_{\phi,n}^{*(m)} = m\sum_{j=1}^{k} \phi\left(\frac{n}{m}(Y_{jm} - Y_{(j-1)m})\right) + (1 - Y_n)\phi^*(0). \tag{4.38}$$

In what follows we would like to deal with statistics $T_\phi^{(m)}$ for more general functions $\phi$ than just $\phi \in \mathbf{\Phi}_{\mathrm{div}}$. Therefore we prefer the more universal statistics (4.36).

In the remaining sections we study the properties and applications of the simple-spacings based variants

$$T_\phi = T_{\phi,n} = \sum_{j=1}^{n-1} \phi\left(n(Y_j - Y_{j-1})\right) + \phi\left(n(1 - Y_{n-1})\right) \tag{4.39}$$

of the statistics $T_\phi^{(1)}$ in (4.36), i.e. $T_\phi^{(m)}$ when $m = 1$. We decompose them into **representative parts** $R_\phi$ and an **asymptotically vanishing parts** $V_\phi$ as follows:

$$T_\phi = R_\phi + V_\phi \quad \text{with} \quad R_\phi = \sum_{j=1}^{n+1} \phi\left(n\left(Y_j - Y_{j-1}\right)\right), \quad \phi \in \mathbf{\Phi} \tag{4.40}$$

and

$$V_\phi = \phi\left(n(1 - Y_{n-1})\right) - \phi\left(n(Y_n - Y_{n-1})\right) - \phi\left(n(1 - Y_n)\right) \tag{4.41}$$

where we put as before $Y_{n+1} = 1$.

**Example 4.2.1** The power divergences $D_\alpha(\boldsymbol{p}_0, \boldsymbol{p}_n)$ of orders $\alpha \in \mathbb{R}$ from Example 3.1.3 define spacings-based statistic $T_\alpha = T_{\alpha,n}$ obtained by inserting in (4.39) the generating power functions $\phi_\alpha$ or $\tilde{\phi}_\alpha$. For example, the power divergence of order 2 with the generating function $\phi_2(t) = (t-1)^2/2$ given in (3.14) defines the spacings-based statistic $T_2 = T_{2,n}$ with the representative part

$$R_2 = R_{2,n} = \frac{1}{2} \sum_{j=1}^{n+1} \left(n(Y_j - Y_{j-1}) - 1\right)^2 = \frac{n^2}{2} \sum_{j=1}^{n+1} \left(Y_j - Y_{j-1} - \frac{1}{n}\right)^2. \tag{4.42}$$

Obviously, this is a spacings-based version of the classical Pearson goodness-of-fit statistic

$$T_n = nD_2(\boldsymbol{p}_n, \boldsymbol{p}_0) = \frac{1}{2}n\chi^2(\boldsymbol{p}_n, \boldsymbol{p}_0) \quad \text{(see (2.16))}.$$

Similarly, the power divergence of order 0 with the generating function $\phi_0(t) = -\ln t$ from (3.15) defines the spacings-based statistic $T_0 = T_{0,n}$ with the representative part

$$R_0 = R_{0,n} = -\sum_{j=1}^{n+1} \ln\left(n\left(Y_j - Y_{j-1}\right)\right), \tag{4.43}$$

which is nothing but a spacings-based version of the classical likelihood ratio goodness-of-fit statistic

$$T_n = nD_0(\boldsymbol{p}_0, \boldsymbol{p}_n) = nD_1(\boldsymbol{p}_n, \boldsymbol{p}_0) \quad \text{(see (3.11))}.$$

# 5  Goodness-of-fit statistics based on spacings

## 5.1  Objectives of the following sections

This chapter is devoted to the systematic analysis of the disparity and divergence spacings statistics and to their comparisons with the spacings statistics studied in the previous literature. The primary aim is to show that while the motivation of the latter is not based on the concept of similarity between empirical and hypothetical distributions, this idea is in fact hidden somewhere behind because they asymptotically coincide with the former. Therefore the *first objective* is to prove the mutual asymptotic equivalence of the disparity and divergence spacings statistics introduced in this chapter and the spacings statistics known from the literature. This equivalence helps to understand why many ad hoc defined spacings-based statistics exhibit desirable asymptotic properties. The secondary aim is to present in a relatively simple unified manner the asymptotic properties of the many various types of spacings statistics studied in the

previous literature. Thus the *second objective* of this chapter is to prove the consistency and asymptotic normality under fixed and local alternatives for a sufficiently wide variety of our spacings-type disparity or divergence statistics. These results are important for applications of the spacings statistics in the testing of goodness-of-fit, and they may also be useful in the estimation of functionals of the type of $\phi$-disparity or $\phi$-divergence. The last aim is to apply this asymptotic theory to the spacings-based power divergence statistics and compare their asymptotic parameters and properties for various divergence orders $\alpha \in \mathbb{R}$. Therefore the *third objective* is the explicit evaluation of the asymptotic parameters of spacings-based power divergence statistics and an analysis of their properties including their continuity in the parameter $\alpha \in \mathbb{R}$. To achieve all these objectives within a reasonably limited space, we pay the main attention to the simple spacings with $m = 1$ and, starting with Section 6, we deal exclusively with simple spacings.

It seems that the spacings-based goodness-of-fit test statistics given in the literature lacked sofar the motivation of taking into account the notion of disparity between hypothetical and empirical distributions $\boldsymbol{p}_0$ and $\boldsymbol{p}_n$. This contrasts with the goodness-of-fit statistics based on deterministic partitions specified by the uniformly distributed constant cutpoints $c_j$ given in (4.21) and by the related random frequency counts (4.23), where the typical statistics, including the most classical Pearson statistic $T_1$ and likelihood ratio statistic $T_0$, can easily be recognized as appropriately scaled power divergences between $\boldsymbol{p}_0$ and $\boldsymbol{p}_n$.

The classical spacings-based statistics, however, appear to have been motivated rather by other considerations such as the analytic simplicity of formulas and the possibility to achieve desired asymptotic properties. In fact, as pointed out by Pyke(1965) in his landmark paper, most of the classical spacings-based statistics were proposed within the context of testing the randomness of events in time, in which differences between successive order statistics (spacings) were considered to play an important role. Also, in the period 1946-1953, when most of the classical tests based on spacings were proposed, research focused mostly on studying the behavior of these tests under the null-hypothesis, rather than under an alternative, making it unnecessary to motivate the test statistic from the point of view of divergence or disparity. Although the concept of dispersion of spacings around the uniform distribution has been mentioned as a motivation for a test statistic by some authors, all known spacings-based statistic are close to the divergence statistic $T_\phi^{(m)}$ of (4.36) or $T_\phi$ of (4.39) for some $\phi$ in $\boldsymbol{\Phi}_{\mathrm{div}}$, but none of them happens to be precisely equal to this divergence statistic. This situation is illustrated in the next examples for the simple-spacings statistics $T_\phi = T_{\phi,n}$ with $\phi \in \boldsymbol{\Phi}$ given by (4.40) as the sum $R_\phi + V_\phi$ where the representative terms

$$R_\phi = \sum_{j=1}^{n+1} \phi\left(n\left(Y_j - Y_{j-1}\right)\right), \quad \phi \in \boldsymbol{\Phi} \tag{5.1}$$

slightly differ from majority of the statistics known from the literature, which are of the form

$$S_\phi = \sum_{j=1}^{n+1} \phi\left((n+1)\left(Y_j - Y_{j-1}\right)\right), \quad \phi \in \boldsymbol{\Phi} \tag{5.2}$$

where $\phi$ is often from the divergence subclass $\boldsymbol{\Phi}_{\mathrm{div}} \subset \boldsymbol{\Phi}$. Hence the departure from the divergence statistics is mainly the scaling of the spacings by $n + 1$ instead of $n$. A possible explanation for this is suggested in the next example. We prove in the next section that the departure from the divergence statistics as such is asymptotically negligible in the sense that the so-called asymptotically vanishing term

$$V_\phi = \phi\left(n(1 - Y_{n-1})\right) - \phi\left(n(Y_n - Y_{n-1})\right) - \phi\left(n(1 - Y_n)\right) \tag{5.3}$$

really vanishes asymptotically and the modification of the scaling factor by $n/(n+1)$ is asymptotically negligible.

**Example 5.1.1** The first known statistic of the type (5.2) is

$$\mathcal{G} = \sum_{j=1}^{n+1} (Y_j - Y_{j-1})^2 \tag{5.4}$$

of Greenwood (1946) who devised it for testing the hypothesis that the intervals between successive events in epidemiology are exponentially distributed. Obviously,

$$(n+1)^2 \mathcal{G} = S_\phi = \sum_{j=1}^{n+1} \phi \left( (n+1) \left( Y_j - Y_{j-1} \right) \right) \tag{5.5}$$

for $\phi(t) = t^2$ from $\boldsymbol{\Phi}$. Therefore the Greenwood proposal was neither the divergence nor the disparity spacing statistic. However, Irwin in the discussion of Greenwood (1946), and Kimball (1947) suggested to replace $\mathcal{G}$ by the modification of the power divergence spacing statistic (4.42) defined by

$$\mathcal{K} = \sum_{j=1}^{n+1} \left( Y_j - Y_{j-1} - \frac{1}{n+1} \right)^2 = \frac{2}{(n+1)^2} \sum_{j=1}^{n+1} \tilde{\phi}_2 \left( (n+1) \left( Y_j - Y_{j-1} \right) \right) \tag{5.6}$$

for $\phi_2(t) = (t-1)^2/2$ from $\boldsymbol{\Phi}_{\mathrm{div}}$ which is the same function that generated the quadratic spacing statistics $R_2$ in (4.42). The motivation of the Irwin and Kimball statistic $\mathcal{K}$ may be deduced from the inequality

$$E \sum_{j=1}^{n+1} \left( Y_j - Y_{j-1} - \frac{1}{n+1} \right)^2 \geq E \sum_{j=1}^{n+1} \left( Y_j - Y_{j-1} - E(Y_j - Y_{j-1}) \right)^2$$

where the inequality is strict unless

$$E \left( Y_j - Y_{j-1} \right) = \frac{1}{n+1} \quad \text{for} \ \ 1 \leq j \leq n+1$$

which in turn takes place if and only if $F = F_0$. Therefore the minimal expected values of the Kimball criterion $\mathcal{K}$ characterize the hypothesis $F_0$ and the larger expected values are reserved for the alternatives $F \neq F_0$.

**Example 5.1.2**   Another classical spacing statistic which is a slight modification of the power divergence spacings statistic (4.43) was defined by Moran (1951) as

$$\mathcal{M} = S_{\phi_0} = -\sum_{j=1}^{n+1} \ln \left( (n+1) \left( Y_j - Y_{j-1} \right) \right) = \sum_{j=1}^{n+1} \phi_0 \left( (n+1) \left( Y_j - Y_{j-1} \right) \right) \tag{5.7}$$

where $\phi_0(t) = -\ln t$ from $\boldsymbol{\Phi}_{\mathrm{div}}$ generated the logarithmic spacing statistics $R_0$ in (4.43). Van Es (1992) considered a statistic $\mathcal{VE}$ which for $m = 1$ takes on the form

$$\mathcal{VE} = \frac{1}{n-1} \sum_{j=2}^{n} \ln \left( (n+1) \left( Y_j - Y_{j-1} \right) \right) = \frac{1}{1-n} S_{\phi_0} + \frac{1}{1-n} \ln \frac{Y_1}{1-Y_n}. \tag{5.8}$$

Hence $(1-n)\mathcal{VE}$ is a modification of (5.7).

## 5.2   Types of statistics studied

As stated above, the analysis of the spacings-based disparity or divergence statistics generated by functions $\phi \in \boldsymbol{\Phi}$ is in the rest of this chapter restricted to the case of the simple spacings with $m = 1$. In the previous subsection we defined for these spacings three different statistics, viz. $T_\phi$ of (4.39), $R_\phi$ of (5.1), and $S_\phi$ of (5.2). The first of these was obtained by application of the partition of the observation space by $n$ empirical percentiles of equidistant orders to the disparity or divergence $D_\phi(F_0, F_n)$ of the hypothetical and empirical distribution. The remaining two were modifications of $T_\phi$ representing the spacing statistics from the pioneering work of Greenwood, Kimball and Moran. In the present subsection some other modifications are introduced, which represent the spacing statistics known from the literature subsequent to the mentioned pioneering work. Since those statistics generally used the $m$-spacings, we

return temporarily in this subsection to our disparity or divergence statistics $T_\phi^{(m)}$ from (4.36) based on $m$-spacings, in order to make the comparisons more realistic and credible.

Let us start with Del Pino's (1979) class of statistics of the form

$$S_\phi^{(m)} = m \sum_{j=1}^{k} \phi\left(\frac{n+1}{m}(Y_{mj} - Y_{m(j-1)})\right) \tag{5.9}$$

where it is assumed that $n+1$ is divisible by $k$ and that $m = (n+1)/k \geq 1$. Here the notation in our chapter is consistent in the sense that (5.9) reduces for $m = 1$ to the formula for $S_\phi$ in (5.2). Del Pino found $\phi(t) = t^2$ to be optimal among the functions $\phi$ considered by him. The class (5.9) was later investigated by Jammalamadaka *et al.* (1989) and many others. Jammalamadaka *et al* studied the asymptotics of $S_\phi^{(m)}$ for $m$ tending slowly to infinity as $n \to \infty$. In such case these asymptotics depend only on the local properties of $\phi(t)$ in the neighborhood of $t = 1$, and in this regard a wide class of functions $\phi$ can be admitted, including those with $\phi''(1) = 0$, so that they can be used for functions which generate disparities or divergences. However, as we have seen in Examples 5.1.1 and 5.1.2 for some $\phi \in \mathbf{\Phi}_{\mathrm{div}}$, the statistics (5.9) differ from those in (4.36). Other examples of well-known spacings-based statistics which differ from our spacings-type $\phi$-disparity statistics (4.36) will be given below. Therefore it is important to look at the problem whether the classical spacings-based statistics and our spacings-type disparity statistics are asymptotically equivalent for $n \to \infty$, and, if yes, then in what precise sense.

Let us return to our spacings-type $\phi$-disparity statistic $T_\phi^{(m)}$ introduced in (4.36). Notice that $T_\phi^{(m)}$ cannot be efficient when $m > 1$, because then it ignores the observations $Y_{mj+r}$ for $1 \leq j \leq k-1$ and $1 \leq r \leq m-1$. Shifting the orders $j/k$ of the percentiles in (4.30) by a quantity depending on $r$, we obtain the shifted empirical percentiles

$$c_j^{(r)} = Q_n\left(\frac{mj+r}{n}\right) = Y_{mj+r}, \quad 1 \leq j \leq k-1, \quad 1 \leq r \leq m-1 \tag{5.10}$$

as cutpoints and, instead of $p_{0j} = Y_{mj} - Y_{m(j-1)} = p_{0j}^{(0)}$, the shifted hypothetical probabilities $p_{0j}^{(r)} = Y_{mj+r} - Y_{m(j-1)+r}$, while still preserving the uniform shifted empirical probabilities $p_{nj}^{(r)} = 1/k = m/n$ on the cells $(c_{j-1}^{(r)}, c_j^{(r)}]$, $1 \leq r \leq m-1$. Replacing each term $\phi(\frac{n}{m}(Y_{mj} - Y_{m(j-1)}))$ in (4.36) by the average

$$\frac{1}{m} \sum_{r=0}^{m-1} \phi\left(\frac{n}{m}(Y_{mj+r} - Y_{m(j-1)+r})\right) \tag{5.11}$$

of all $\phi(np_{0j}^{(r)}/m)$ for $0 \leq r \leq m-1$, we get a potentially more efficient version of $T_\phi^{(m)}$, namely

$$\hat{T}_\phi^{(m)} = \sum_{j=m}^{n-1} \phi\left(\frac{n}{m}(Y_j - Y_{j-m})\right) + m\phi\left(\frac{n}{m}(1 - Y_{n-m})\right) \tag{5.12}$$

which for $m = 1$ reduces to $T_\phi$ of (4.39), so that the notation of our chapter is again consistent.

A similar procedure can be carried out for $S_\phi^{(m)}$ of (5.9), which involves the observations $Y_{mj}$ for $1 \leq j \leq k$, but ignores the observations $Y_{mj+r}$ for $0 \leq j \leq k-1$ and $1 \leq r \leq m-1$. A similar procedure can be carried out for $S_\phi^{(m)}$ of (5.9), which involves the observations $Y_{mj}$ for $1 \leq j \leq k$, but ignores the observations $Y_{mj+r}$ for $0 \leq j \leq k-1$ and $1 \leq r \leq m-1$. Replacing each term $\phi(\frac{n+1}{m}(Y_{mj} - Y_{m(j-1)}))$ in (5.9) by the average of the $m$ terms $\phi(\frac{n+1}{m}(Y_{mj+r} - Y_{m(j-1)+r}))$, $0 \leq r \leq m-1$, and excluding the terms containing undefined expressions (that is, the terms $Y_{mk+r} - Y_{m(k-1)+r}, 1 \leq r \leq m-1$, where $mk + r > n+1$ ), we get a similar possibly more efficient version

$$\hat{S}_\phi^{(m)} = \sum_{j=m}^{n+1} \phi\left(\frac{n+1}{m}(Y_j - Y_{j-m})\right) \tag{5.13}$$

of Del Pino's statistic $S_\phi^{(m)}$ of (5.9). Notice that if $m = 1$, then $\hat{S}_\phi^{(m)}$ of (5.13) reduces to $S_\phi$ of (5.2) above, so that our notation is in this sense still consistent. The statistics (5.13) are formally well defined for all $1 \leq m \leq n$, and not only for $m = (n+1)/k \geq 1$ corresponding to the integers $1 < k \leq n+1$. Both Del Pino's statistic $S_\phi^{(m)}$ of (5.9) and $\hat{S}_\phi^{(m)}$ of (5.13) can be viewed as generalizations of $S_\phi$ of (5.2) from the case of simple spacings to that of $m$-spacings, but whereas (5.9) involves *non-overlapping* $m$-spacings, (5.13) uses $m$-th order *overlapping* spacings. Cressie (1976, 1979) and Ekström (1999) are among the authors dealing with the statistics (5.13) for fixed $m \geq 1$ and eventually also for $m$ slowly tending to $\infty$ when $n \to \infty$.

In spite of the fact that, when carrying out our analysis above, we went through several important papers (and many other ones listed in these as references), covering altogether four decades of research on spacings-based statistics, we did not in this literature come across the framework of the statistics $T_\phi$ and its modification $R_\phi$ when restricting ourselves to the simple spacings with $m = 1$. To make this connection, take into account that if $m > 1$, and in particular if $m \to \infty$, then the statistics (5.13) assign more weight to the central spacings than to those in the tails. To avoid this, Hall (1986) proposed to wrap the observations $Y_1, Y_2, ..., Y_n$ around the circle of unit circumference and to define the $m$-spacings $Y_{m+j} - Y_j$ for arbitrary $1 \leq m \leq n$ and $j$ as the distance between observations $Y_j$ and $Y_{j+m}$ on this circle. This leads to the following two possible extensions of the ordered observations $Y_1, \ldots, Y_n$.

**(i)** By the formula

$$Y_{n+j} = 1 + Y_j \quad \text{for} \quad j = 1, 2, ..., n \tag{5.14}$$

where the previous dummy observation $Y_0 = 0$ is suppressed and the other dummy observation $Y_{n+1} = 1$ is redefined in accordance with (5.14) by $Y_{n+1} = 1 + Y_1$, leading to the $m$-spacing $Y_{j+m} - Y_j$ to be equal to $1 + Y_{m+j-n} - Y_j$ if $n + 1 - m \leq j \leq n$, and remaining unaltered, i.e. equal to $Y_{j+m} - Y_j$, if $1 \leq j \leq n - m$.

**(ii)** By the alternative formula

$$Y_{n+j} = 1 + Y_{j-1} \quad \text{for} \quad j = 0, 1, \cdots, n \tag{5.15}$$

where the dummy observations $Y_0 = 0$ and $Y_{n+1} = 1$ are placed on the circle as well, resulting in the $m$-spacing $Y_{j+m} - Y_j$ to be defined as $1 + Y_{m+j-n-1} - Y_j$ if $n + 2 - m \leq j \leq n$ and remaining equal to $Y_{j+m} - Y_j$ if $0 \leq j \leq n + 1 - m$.

Both these extensions of the ordered observations $Y_j$ beyond $j > n$ allow to add in (5.13) the tail evidence missing there by adding to the substituted averages (5.11) also the previously excluded terms. Depending on whether we use (5.14) or the alternative extension (5.15), we get in this manner two different extensions of (5.13), namely

$$\tilde{S}_\phi^{(m)} = \sum_{j=1}^n \phi\left((n+1)\frac{Y_{j+m} - Y_j}{m}\right) \quad \text{where} \quad Y_{j+m} = 1 + Y_{j+m-n} \tag{5.16}$$

if $j = n + 1 - m, \cdots, n$ (c.f. (5.14)), or

$$\tilde{\tilde{S}}_\phi^{(m)} = \sum_{j=0}^n \phi\left((n+1)\frac{Y_{j+m} - Y_j}{m}\right) \quad \text{where} \quad Y_{j+m} = 1 + Y_{j+m-n-1} \tag{5.17}$$

if $j = n + 2 - m, \cdots, n$, $Y_0 = 0$ and $Y_{n+1} = 1$ (c.f. (5.15)).

The statistics from the class (5.16) were studied for example by Hall (1986) and Morales *et al.* (2003), while those from the class (5.17) were investigated among others by Cressie (1978), Rao and Kuo (1984), Ekström (1999) and Misra and van der Meulen (2001) and others cited there.

Recently Jimenez and Shao (2009) studied for convex functions $\phi$ the statistics

$$JS_\phi^{(m)} = m \sum_{j=1}^k \phi\left(\frac{n+1}{m}\left[F(Y_{mj}) - F(Y_{m(j-1)})\right]\right)$$

for $m, n$ such that $n+1$ is divisible by $k$ and $m = (n+1)/k \geq 1$. Under the hypothetic p.d.f.'s $F(y) = y$ considered in this chapter these statistics reduce to the statistic $S_\phi^{(m)}$ of Del Pino's (5.9).

As said above, this chapter deals only with the ordinary spacings where $m = 1$. We have seen that then the statistic $T_\phi$ takes on the form presented in (4.39) and both $S_\phi^{(m)}$ of (5.9) and $\hat{S}_\phi^{(m)}$ of (5.13) reduce to the statistic

$$S_\phi = \sum_{j=1}^{n+1} \phi\left((n+1)\left(Y_j - Y_{j-1}\right)\right), \quad \text{where } Y_{n+1} = 1 \text{ and } Y_0 = 0 \tag{5.18}$$

introduced in (5.2). Consequently, $JS_\phi^{(m)}$ reduces for $m = 1$ to $S_\phi$ too. It is easy to see that in this case also $\tilde{\hat{S}}_\phi^{(m)}$ of (5.17) reduces to $S_\phi$. However, $\tilde{S}_\phi^{(m)}$ of (5.16) does not do so unless $\phi$ is linear. Indeed, if $m = 1$, $\tilde{S}_\phi^{(m)}$ reduces to

$$\tilde{S}_\phi = \sum_{j=1}^{n-1} \phi\left((n+1)\left(Y_{j+1} - Y_j\right)\right) + \phi\left((n+1)\left(Y_1 + 1 - Y_n\right)\right) \tag{5.19}$$

which coincides with

$$S_\phi = \sum_{j=1}^{n-1} \phi\left((n+1)\left(Y_{j+1} - Y_j\right)\right) + \phi\left((n+1)Y_1\right) + \phi\left((n+1)\left(1 - Y_n\right)\right) \tag{5.20}$$

only if

$$\phi\left((n+1)Y_1\right) + \phi\left((n+1)\left(1 - Y_n\right)\right) = \phi\left((n+1)\left(Y_1 + 1 - Y_n\right)\right)$$

which takes place with a positive probability only for linear $\phi$.

In addition to the statistics $T_\phi$, $S_\phi$, $R_\phi$ and $\tilde{S}_\phi$ introduced above, we will study two more single-spacings statistics. First, we introduce another new type of spacings-type disparity statistic denoted by $\tilde{T}_\phi$. To obtain it, we redefine the partition $\mathcal{Q} = \{(0, Y_1], \ldots, (Y_{n-2}, Y_{n-1}], (Y_{n-1}, 1)\}$ of $(0, 1)$ considered in (4.33). The new partition $\tilde{\mathcal{Q}}$ of $(0, 1)$ is obtained by rearranging the $n$ intervals of the partition $\mathcal{Q}$ into $n$ new intervals by the rule

$$(0, Y_1] \mapsto (0, Y_1] \cup (Y_n, 1) \quad \text{and} \quad (Y_{n-1}, 1) \mapsto (Y_{n-1}, Y_n] \tag{5.21}$$

while keeping the remaining intervals $(Y_{j-1}, Y_j]$, $2 \leq j \leq n-1$ unaltered. This new partition $\tilde{\mathcal{Q}}$ leads to the **modified hypothetical distribution**

$$\tilde{\boldsymbol{p}}_0 = (\tilde{p}_{01} = Y_1 + 1 - Y_n, \tilde{p}_{02} = Y_2 - Y_1, \ldots, \tilde{p}_{0n} = Y_n - Y_{n-1})$$

which similar to (4.32) preserves the original uniform empirical distribution $\boldsymbol{p}_n$ on the cells, as each of the new $n$ intervals still contains exactly one of the observations $Y_1, \ldots, Y_n$. Therefore the new partition leads to the new spacings-type disparity statistics

$$\begin{aligned} \tilde{T}_\phi &= n\, D_\phi\left(\tilde{\boldsymbol{p}}_0, \boldsymbol{p}_n\right) = \sum_{j=1}^{n} \phi(n\tilde{p}_{0j}) \\ &= \sum_{j=2}^{n} \phi\left(n(Y_j - Y_{j-1})\right) + \phi\left(n(Y_1 + 1 - Y_n)\right) \end{aligned} \tag{5.22}$$

which differ from $T_\phi$ of (4.39). Applying the functions

$$\phi^{(n)}(t) = \phi\left(\frac{n+1}{n}t\right) \tag{5.23}$$

we obtain the useful relations

$$\tilde{S}_\phi = \tilde{T}_{\phi^{(n)}} \quad \text{and} \quad S_\phi = R_{\phi^{(n)}}. \tag{5.24}$$

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In addition to the statistics $R_\phi, S_\phi, \tilde{S}_\phi, T_\phi, \tilde{T}_\phi$, defined above in (5.1), (5.2), (5.19), (4.39), and (5.22), respectively, we use in this chapter also the auxiliary spacings-based statistics

$$\tilde{R}_\phi = \sum_{j=1}^{n-1} \phi\left(n(Y_{j+1} - Y_j)\right) = R_\phi - \phi(nY_1) - \phi\left(n(1 - Y_n)\right) \tag{5.25}$$

investigated previously by authors neglecting the tail probabilities such as for example Hall (1984). Thus we can conclude this subsection by introducing the sets

$$\mathcal{U}_\phi = \left\{ R_\phi, \tilde{R}_\phi, S_\phi, \tilde{S}_\phi, T_\phi, \tilde{T}_\phi \right\}, \quad \phi \in \mathbf{\Phi} \tag{5.26}$$

of the spacings-based statistics of the special types introduced here and studied in the following sections. The statistics from $\mathcal{U}_\phi$ are representative in the sense that they cover almost all spacings based statistics studied in the previous literature as special cases when the attention is restricted to the simple spacings. The only exception known to us are the statistics

$$\bar{R}_\phi = \tilde{R}_{\phi_n} = \tilde{S}_\phi - \phi\left((n + 1)(Y_1 + 1 - Y_n)\right) \tag{5.27}$$

studied by van Es (1992), see e.g. (5.8). However, it is seen from the proof of Assertion 6.1.1 below that the asymptotic properties of $\bar{R}_\phi$ are the same as the asymptotic properties of the statistics from the class $\mathcal{U}_\phi$ studied in the rest of this chapter.

The set $\mathbf{\Phi}$ of differentiable functions $\phi : (0, \infty) \to \mathbb{R}$ was introduced in the Subsection 3.1 by mild additional conditions guaranteeing the existence of the integrals

$$D_\phi(P, P_0) = \int p_0 \phi\left(\frac{p}{p_0}\right) d\lambda \quad \text{(cf. (3.7))}$$

called disparities in the wide sense, which were justified as measures of disparity, divergence or distance only for $\phi$ from the subclasses $\mathbf{\Phi}_{\text{disp}} \supset \mathbf{\Phi}_{\text{div}}$ of $\mathbf{\Phi}$. On the other hand, the functions $\phi$ which defined the statistics $U_\phi \in \mathcal{U}_\phi$ considered in the cited literature imposed on the functions $\phi : (0, \infty) \to \mathbb{R}$ usually somewhat different additional conditions, namely the continuity and the continuous second order differentiability in the neighborhood of 1 with $\phi''(1) \neq 0$ and $\phi(1) = 0$. Therefore in the sequel we study the class of statistics

$$\mathcal{U}_\phi = \left\{ R_\phi, \tilde{R}_\phi, S_\phi, \tilde{S}_\phi, T_\phi, \tilde{T}_\phi \right\}, \quad \phi \in \mathbf{\Phi}_0 \tag{5.28}$$

where $\mathbf{\Phi}_0$ is the set of all continuous functions $\phi : (0, \infty) \mapsto R$ which are twice continuously differentiable in a neighborhood of 1 with $\phi''(1) > 0$ and $\phi(1) = 0$. The replacement of $\mathbf{\Phi}$ by $\mathbf{\Phi}_0$ means no loss of generality and guarantees that the class (5.28) contains all disparity and divergence statistics $R_\phi, S_\phi, \tilde{S}_\phi, T_\phi, \tilde{T}_\phi$, defined above or, more generally, that

$$\mathbf{\Phi}_{\text{div}} \subset \mathbf{\Phi}_{\text{disp}} \subset \mathbf{\Phi}_0. \tag{5.29}$$

For references later, we summarize the definition formulas

$$R_\phi = R_{\phi,n} = \sum_{j=1}^{n+1} \phi\left(n\left(Y_j - Y_{j-1}\right)\right) \tag{5.30}$$

$$S_\phi = S_{\phi,n} = \sum_{j=1}^{n+1} \phi\left((n + 1)\left(Y_j - Y_{j-1}\right)\right) \tag{5.31}$$

(e.g. Jammalamadaka et al. (1989), Jiménez and Shao (2009))

$$T_\phi = T_{\phi,n} = \sum_{j=1}^{n-1} \phi\left(n(Y_j - Y_{j-1})\right) + \phi\left(n(1 - Y_{n-1})\right) = nD_\phi(\boldsymbol{p}_0, \boldsymbol{p}_n) \quad \text{(cf. (4.36), (4.39))} \tag{5.32}$$

$$\tilde{R}_\phi = \tilde{R}_{\phi,n} = \sum_{j=2}^{n} \phi\left(n(Y_j - Y_{j-1})\right) \tag{5.33}$$

(e.g. Hall (1984), Cressie (1976) - (1979))

$$\tilde{S}_\phi = \tilde{S}_{\phi,n} = \sum_{j=2}^{n} \phi\left((n+1)\left(Y_j - Y_{j-1}\right)\right) + \phi\left((n+1)\left(Y_1 + 1 - Y_n\right)\right) \tag{5.34}$$

(e.g. Hall (1986), Morales et al. (2003))

$$\tilde{T}_\phi = \tilde{T}_{\phi,n} = \sum_{j=2}^{n} \phi\left(n(Y_j - Y_{j-1})\right) + \phi\left(n(Y_1 + 1 - Y_n)\right) = nD_\phi(\tilde{\boldsymbol{p}}_0, \boldsymbol{p}_n) \quad \text{(cf. (5.22))} \tag{5.35}$$

where in all these formulas $Y_0 = 0, Y_{n+1} = 1$ and

$$Y_j = F_0(X_{n:j}) \quad \text{for } X_{n:j} \sim F, \ 1 \le j \le n \quad \text{and } \mathcal{H} : F_0. \tag{5.36}$$

In the next subsection and in the rest of the chapter we present a series of assertions about the properties of the statistics (5.30) - (5.35). Some of these assertions follow from the papers dealing with general $m$-spacing statistics cited in this and the following sections. Our simple proofs are to some extent based on the arguments established there.

## 5.3  Structural spacings statistics

In this subsection and in the rest of this chapter we study the subclasses

$$\mathcal{U}_\phi = \left\{ R_\phi, \tilde{R}_\phi, S_\phi, \tilde{S}_\phi, T_\phi, \tilde{T}_\phi \right\} \quad \text{for } \phi \in \boldsymbol{\Phi}_2 \ \text{ or } \phi \in \boldsymbol{\Phi}_1 \quad \text{where } \boldsymbol{\Phi}_2 \subset \boldsymbol{\Phi}_1 \subset \boldsymbol{\Phi}_0 \tag{5.37}$$

of the class (5.28) of statistics defined by (5.30) - (5.36). Here $\boldsymbol{\Phi}_1 \subset \boldsymbol{\Phi}_0$ is the subset of functions $\phi \in \boldsymbol{\Phi}_0$ for which there exist functions $\xi, \eta, \zeta : (0, \infty) \mapsto \mathbb{R}$ satisfying the functional equation

$$\phi(st) = \xi(s)\,\phi(t) + \zeta(t)\,\phi(s) + \eta(s)\,(t-1) \quad \text{for all } s, t \in (0, \infty). \tag{5.38}$$

The narrower class $\boldsymbol{\Phi}_2$ consists of all $\phi \in \boldsymbol{\Phi}_1$ which admit functions $\xi, \eta, \zeta : (0, \infty) \mapsto \mathbb{R}$ satisfying the stronger functional equation

$$\phi(st) = \xi(s)\,\phi(t) + \phi(s) + \eta(s)\,(t-1). \tag{5.39}$$

**Assertion 5.3.1**  The functions $\xi$, $\zeta$ and $\eta$ are continuous on $(0, \infty)$ and satisfy the relations

$$\xi(1) = \zeta(1) = 1 \quad \text{and} \quad \eta(1) = 0. \tag{5.40}$$

**Proof**  The continuity of $\xi$ and $\eta$ can be obtained by putting $t = 2$ and $t = 3$, and that of $\zeta$ by putting $s = 2$ in (5.38). If we put $s = 1$ in (5.38) or (5.39) and use the assumption $\phi(1) = 0$, then we obtain that for all $t \in (0, \infty)$

$$(\xi(1) - 1)\,\phi(t) + \eta(1)\,(t - 1) = 0.$$

This contradicts the assumption $\phi''(1) > 0$, unless $\xi(1) = 1$ which implies also $\eta(1) = 0$. By putting $t = 1$ in (5.38) we find that $\zeta(1) = 1$. $\qquad\square$

**Assertion 5.3.2**  Every $\phi \in \boldsymbol{\Phi}_1$ is differentiable on $(0, \infty)$, the corresponding functions $\xi$ and $\eta$ are differentiable at 1, and for every $t > 0$

$$\phi'(t) = \xi'(1)\,\frac{\phi(t)}{t} + \phi'(1)\,\frac{\zeta(t)}{t} + \eta'(1)\,\frac{t-1}{t}. \tag{5.41}$$

**Proof** Putting $s = 1 + \varepsilon$ and

$$\xi^*(\varepsilon) = \frac{\xi(1 + \varepsilon) - \xi(1)}{\varepsilon}, \quad \eta^*(\varepsilon) = \frac{\eta(1 + \varepsilon) - \eta(1)}{\varepsilon}$$

we obtain from (5.38) for every $t > 0$ and $\varepsilon$ close to 0

$$t \frac{\phi(t + \varepsilon t) - \phi(t)}{\varepsilon t} = \xi^*(\varepsilon)\, \phi(t) + \frac{\phi(1 + \varepsilon) - \phi(1)}{\varepsilon}\, \zeta(t) + \eta^*(\varepsilon)\,(t - 1). \tag{5.42}$$

Since $\phi$ is differentiable in a neighborhood of 1, we have for $t$ close to 1

$$\xi^*(\varepsilon)\, \phi(t) + \eta^*(\varepsilon)\,(t - 1) = t\, \phi'(t) - \phi'(1)\, \zeta(t) + o(\varepsilon) \quad \text{as } \varepsilon \to 0.$$

By the assumptions concerning $\boldsymbol{\Phi}$, $\phi(t)$ is not linear in a neighborhood of $t = 1$. Therefore the last relation implies that the limits of $\xi^*(\varepsilon)$ and $\eta^*(\varepsilon)$ for $\varepsilon \to 0$ exist, that is,

$$\xi^*(\varepsilon) = \xi'(1) + o(\varepsilon) \quad \text{and} \quad \eta^*(\varepsilon) = \eta'(1) + o(\varepsilon) \quad \text{as } \varepsilon \to 0.$$

Now (5.41) follows from (5.42) for all $t > 0$. $\qquad\square$

**Example 5.3.1** The function $\phi(t) = (1 - t)/t$, $t > 0$, belongs to $\boldsymbol{\Phi}$ and satisfies (5.39) for $\xi(t) = 1/t$ and $\eta(t) = 0$. Therefore it belongs to $\boldsymbol{\Phi}_2 \subset \boldsymbol{\Phi}$. The function $\phi(t) = (1 - t)^2/t$, $t > 0$, belongs to $\boldsymbol{\Phi}$ too and satisfies (5.39) for the same $\xi(t)$ as above and $\eta(t) = t - 1/t$. Therefore it belongs to $\boldsymbol{\Phi}_2$. The functions defined on $(0, \infty)$ by

$$\phi_\alpha(t) = \frac{t^\alpha \ln t}{(2\alpha - 1)}, \quad \alpha \in \mathbb{R} - \{\tfrac{1}{2}\}$$

belong to $\boldsymbol{\Phi}$ and satisfy (5.38) for $\xi(t) = \zeta(t) = t^\alpha$ and $\eta(t) = 0$. Therefore

$$\{\phi_\alpha : \alpha \in \mathbb{R} - \{\tfrac{1}{2}\}\} \subset \boldsymbol{\Phi}_1$$

and $\phi_0 \in \boldsymbol{\Phi}_2$. But $\phi_1$ satisfies also (5.39) for $\xi(t) = t$ and $\eta(t) = t \ln t$ and therefore $\phi_1$ belongs to $\boldsymbol{\Phi}_2$.

## 5.4 Organization of the following sections

The rest of the chapter deals with the asymptotic properties and applications of the classes of statistics considered in (5.37). Let us mention briefly how the following sections are organized. Section 6 establishes the asymptotic equivalence of the statistics from the class $\mathcal{U}_\phi = \left\{ R_\phi, \tilde{R}_\phi, S_\phi, \tilde{S}_\phi, T_\phi, \tilde{T}_\phi \right\}$, $\phi \in \boldsymbol{\Phi}_1$, and presents the general asymptotic theory of the structural statistics from the class $\mathcal{U}_\phi = \left\{ R_\phi, \tilde{R}_\phi, S_\phi, \tilde{S}_\phi, T_\phi, \tilde{T}_\phi \right\}$, $\phi \in \boldsymbol{\Phi}_2$. Section 7 applies this theory to and makes it precise for the power divergence statistics of Example 3.1.3 and comments on comparable results in the previous literature. Section 8 presents a universal program for evaluation of power divergence spacings statistics and their applications in testing the goodness-of-fit. Finally, Section 9 contains proofs of the assertions of Sections 6 and 7.

# 6 Asymptotic properties of structural statistics

In the remainder of this chapter the observations are assumed to be distributed on $(0, 1]$ in two possible ways:

   (i) *under a fixed alternative,*

   (ii) *under local alternatives.*

Case (i) means that the observations are distributed by a fixed distribution function $F \sim f$ with $f$ positive and continuous on $[0, 1]$. Case (ii) means that the observations from samples of sizes $n = 1, 2, \ldots$ are distributed by distribution functions

$$F^{(n)}(x) = F_0(x) + \frac{L_n(x)}{\sqrt[4]{n}} = x + \frac{L_n(x)}{\sqrt[4]{n}} \tag{6.1}$$

on $[0, 1]$, where the functions $L_n : \mathbb{R} \mapsto \mathbb{R}$ are continuously differentiable, with $L_n(0) = L_n(1) = 0$, and with derivatives $\ell_n(x) = L_n'(x)$ tending on $[0, 1]$ to a continuously differentiable function $\ell : \mathbb{R} \mapsto \mathbb{R}$ uniformly in the sense that

$$\sup_{0 \leq x \leq 1} |\ell_n(x) - \ell(x)| = o(1) \quad \text{as } n \to \infty. \tag{6.2}$$

The two possibilities (i) and (ii) are not mutually exclusive: their conjunction is "under the hypothesis $\mathcal{H}_0$" where $F(x) = F_0(x)$, $f(x) = f_0(x) = \boldsymbol{I}_{[0,1]}(x)$ and $L_n(x) = 0$ on $\mathbb{R}$ for all $n$. This means that the asymptotic results obtained under local alternatives for $\ell(x)$ of (6.2) being identically equal to 0 must coincide with the results obtained under the fixed alternative for $F(x) = F_0(x)$.

## 6.1 Asymptotic equivalence

The theorems below demonstrate that if $\phi \in \boldsymbol{\Phi}_2$ defines a $\phi$-divergence or $\phi$-disparity, then the statistics $S_\phi$, $\tilde{S}_\phi$, $R_\phi$ and $\tilde{R}_\phi$, which are formally not scaled $\phi$-divergences or $\phi$-disparities of the hypothetical and empirical distributions $F_0$ and $F_n$, share the most important statistical properties with the statistics $T_\phi$ and $\tilde{T}_\phi$, which are scaled $\phi$-divergences or $\phi$-disparities of this type . Therefore they provide a key argument for the thesis of the present chapter formulated in Section 2, that the spacings-based goodness-of-fit statistics considered in the previous literature actually measure a disparity between the hypothetical and empirical distributions $F_0$ and $F_n$, although this was possibly not so intended by the various authors. But the main purpose of the following theorems is to present a systematic asymptotic theory for the whole set of statistics (5.37) and to demonstrate that the small modifications distinguishing these statistics from one another are asymptotically negligible. The restriction to the functions from $\boldsymbol{\Phi}_2$ or even $\boldsymbol{\Phi}_1$ is not essential – it only simplifies the proof of the next theorem.

**Assertion 6.1.1.** Consider the observations under fixed or local alternatives, and the set of statistics $\{R_\phi, \tilde{R}_\phi, S_\phi, \tilde{S}_\phi, T_\phi, \tilde{T}_\phi\}$ defined in (5.1), (5.25), (5.2), (5.19), (4.39), and (5.22). If $\phi \in \boldsymbol{\Phi}_1$, then for any statistic $U_\phi \in \{R_\phi, S_\phi, \tilde{S}_\phi, T_\phi\}$

$$U_\phi - \tilde{R}_\phi = O_p(1) \quad \text{as } n \to \infty, \tag{6.3}$$

and, if $\phi \in \boldsymbol{\Phi}_2$, then

$$S_\phi - R_\phi = \varepsilon_n R_\phi + \delta_n \quad \text{and} \quad \tilde{S}_\phi - \tilde{T}_\phi = \varepsilon_n \tilde{T}_\phi + \delta_n \tag{6.4}$$

where $\varepsilon_n = o(1)$ and $\delta_n = \phi'(1) + o(1)$ as $n \to \infty$. $\qquad\qquad \square$

The proofs of this assertion and of the remaining ones of the chapter are deferred to Section 9.

## 6.2 Assumptions and notations

In this subsection we study the same spacings-type $\phi$-disparity statistics $R_\phi, \tilde{R}_\phi, S_\phi, \tilde{S}_\phi, T_\phi$ and $\tilde{T}_\phi$, defined by (5.1), (5.25), (5.2), (5.19), (4.39), and (5.22), for $\phi$ from $\boldsymbol{\Phi}_2$ or $\boldsymbol{\Phi}_1$ as in the previous subsection. Unless otherwise explicitly stated, these statistics are assumed to be distributed under the fixed or local alternatives introduced as case (i) and case (ii) in the beginning of this section .

For every continuous function $\psi : (0, \infty) \mapsto \mathbb{R}$ we define the condition

$$\lim_{t \to \infty} t^{-\alpha} |\psi(t)| = \lim_{t \downarrow 0} t^\beta |\psi(t)| = 0 \quad \text{for some } \alpha \geq 0 \text{ and } \beta < 1 \tag{6.5}$$

and the integral

$$\langle \psi \rangle = \langle \psi(t) \rangle = \int_0^\infty \psi(t) \, e^{-t} \, \mathrm{d}t. \tag{6.6}$$

Obviously, if (6.5) holds then $\langle \psi \rangle$ exists and is finite.

Let $\phi \in \mathbf{\Phi}_1$ satisfy (6.5) and let

$$\xi = \xi_\phi, \quad \zeta = \zeta_\phi \quad \text{and} \quad \eta = \eta_\phi \tag{6.7}$$

be the corresponding functions satisfying the functional equation (5.38). Then all functions

$$\psi(t) = \phi(ts) - \phi(t)\,\zeta(s), \quad s > 0,$$

satisfy (6.5) too, and by (5.38) the linear combinations

$$\psi(t) = \xi(t)\,\phi(s) + \eta(t)\,(s-1), \quad s > 0,$$

of functions $\xi(t)$ and $\eta(t)$ also satisfy (6.5). Since $\phi(s)$ is not linear in the neighborhood of $s = 1$, it follows from here that $\xi(t)$ and $\eta(t)$ themselves satisfy (6.5). Therefore the integrals $\langle \xi \rangle$ and $\langle \eta \rangle$ exist and are finite.

For the fixed alternatives $F \sim f$ we shall consider the linear combinations

$$\mu_\phi(f) = \langle \xi \rangle\, D_\phi(F_0, F) + \langle \phi \rangle\, D_\zeta(F_0, F) \tag{6.8}$$

where

$$D_\phi(F_0, F) = \int_0^1 f(x)\phi\left(\frac{1}{f(x)}\right) \mathrm{d}x \tag{6.9}$$

and

$$D_\zeta(F_0, F) = \int_0^1 f(x)\zeta\left(\frac{1}{f(x)}\right) \mathrm{d}x \tag{6.10}$$

are disparities of the distributions $F_0$ and $F$, well defined by (4.1) under the present assumptions about the densities $f_0$ and $f$, and are finite. If $\phi(t)$ is convex on $(0, \infty)$, or $\phi(t) - \phi'(1)\,(t-1)$ is monotone on $(0, 1)$ and $(1, \infty)$, then $D_\phi(F_0, F)$ is a nonnegative $\phi$-divergence or $\phi$-disparity of $F_0$ and $F$. Similarly, if $\zeta(t)$ is convex on $(0, \infty)$, or $\zeta(t) - \zeta(1) - \zeta'(1)\,(t-1)$ is monotone on $(0, 1)$ and $(1, \infty)$, then the $\phi^*$-divergence or $\phi^*$-disparity of $F_0$ and $F$ for

$$\phi^*(t) = \zeta(t) - \zeta(1) = \zeta(t) - 1 \quad (\text{cf } (5.40))$$

satisfies the relation $D_{\phi^*}(F_0, F) = D_\zeta(F_0, F) - 1$. Hence the formula for $\mu_\phi(f)$ can be written for every $\phi \in \mathbf{\Phi}_1$ in the more intuitive form

$$\mu_\phi(f) = \langle \xi \rangle\, D_\phi(F_0, F) + \langle \phi \rangle\, [D_{\phi^*}(F_0, F) + 1] \tag{6.11}$$

where $\xi$ and $\phi^*$ depend on $\phi$ as specified above, and $D_\phi(F_0, F)$, $D_{\phi^*}(F_0, F)$ are divergences or disparities between the hypothesis $F_0$ and the alternative $F$ for typical $\phi \in \mathbf{\Phi}_1$. For $\phi \in \mathbf{\Phi}_2 \subset \mathbf{\Phi}_1$ it holds that $\zeta = 1$, so that (6.11) then simplifies to

$$\mu_\phi(f) = \langle \xi \rangle\, D_\phi(F_0, F) + \langle \phi \rangle. \tag{6.12}$$

In particular for $\phi \in \mathbf{\Phi}_2$

$$\mu_\phi(f_0) = \langle \phi \rangle. \tag{6.13}$$

## 6.3   Consistency under hypotheses and fixed alternatives

The consistency results of this section are in fact laws of large numbers for the spacings based disparity and divergence statistics. They are used in the following sections dealing with asymptotic normality under local and fixed alternatives. Note that under fixed alternatives they provide tools for nonparametric estimates of disparities or divergences. Such tools were used e.g. for nonparametric estimation of entropy by Vasicek (1976) or by Dudewicz and van der Meulen (1987). Our main result is formulated for fixed alternatives.

**Assertion 6.3.1**  Consider the observations under a fixed alternative $F \sim f$ with $f$ positive and continuous on $[0,1]$, and denote by $U_\phi$ any statistic from the class $\{R_\phi, \tilde{R}_\phi, T_\phi, \tilde{T}_\phi\}$. If $\phi \in \mathbf{\Phi}_1$ satisfies (6.5), then

$$\frac{U_\phi}{n} \xrightarrow{p} \mu_\phi(f) \quad \text{for } n \to \infty \tag{6.14}$$

where $\mu_\phi(f)$ is given by (6.11). If $\phi \in \mathbf{\Phi}_2$ satisfies (6.5), then the asymptotic relation (6.14) remains valid also for $U_\phi = \tilde{S}_\phi$ and $U_\phi = S_\phi$, and $\mu_\phi(f)$ is given by the simpler formula (6.12).  $\square$

**Corollary 6.3.1**  Under $\mathcal{H}_0$ (6.14)) reduces to

$$\frac{U_\phi}{n} \xrightarrow{p} \mu_\phi(f_0) = \langle \phi \rangle = \int_0^\infty \phi(t)\, e^{-t}\, \mathrm{d}t \qquad \text{for } n \to \infty \tag{6.15}$$

$\square$

In the sequel we use the $L_2$-norm

$$\|\ell\| = \left( \int_0^1 \ell^2(x)\, \mathrm{d}x \right)^{1/2}$$

and we denote the integral (6.6) usually by $\langle \psi(t) \rangle$ rather than $\langle \psi \rangle$.

## 6.4  Asymptotic normality under local alternatives

**Assertion 6.4.1**  Consider the observations under the local alternatives (6.1) with a limit function $\ell(x)$ introduced in (6.2), and denote by $U_\phi$ any statistic from the set $\{R_\phi, \tilde{R}_\phi, S_\phi, \tilde{S}_\phi, T_\phi, \tilde{T}_\phi\}$. If $\phi \in \mathbf{\Phi}_2$ satisfies the stronger version of (6.5) with $\beta < 1/2$ then

$$\frac{1}{\sqrt{n}}(U_\phi - n\mu_\phi) \xrightarrow{\mathcal{D}} N(m_\phi(\ell), \sigma_\phi^2) \quad \text{as } n \to \infty \tag{6.16}$$

where

$$\mu_\phi = \langle \phi(t) \rangle, \quad \sigma_\phi^2 = \langle \phi^2(t) \rangle - \langle \phi(t) \rangle^2 - \left( \langle t\phi(t) \rangle - \langle \phi(t) \rangle \right)^2 \tag{6.17}$$

and

$$m_\phi(\ell) = \frac{\|\ell\|^2}{2} \left( \langle t^2\phi(t) \rangle - 4\langle t\phi(t) \rangle + 2\langle \phi(t) \rangle \right). \tag{6.18}$$

$\square$

## 6.5  Asymptotic normality under fixed alternatives

Let us now consider the fixed alternative $F \sim f$ defined at the beginning of this section under (i), and $\phi \in \mathbf{\Phi}_2$ with $\xi = \xi_\phi$, $\eta = \eta_\phi$, satisfying the functional equation (5.39), and denote by $\phi', \xi', \eta'$ the derivatives of $\phi, \xi, \eta$ as in Assertion 5.3.2. To express the asymptotic normality under this alternative, we need auxiliary functions $\Psi_i = \Psi_{i,\phi}$ of the variable $x \in (0,1)$:

$$
\begin{aligned}
\Psi_1(x) = {} & \xi'(1)\,\langle \phi(t) \rangle\, f(x)\, \xi\left( \frac{1}{f(x)} \right) + \xi'(1)\, f(x)\, \phi\left( \frac{1}{f(x)} \right) \\
& + \left[ \phi'(1) - \eta'(1) \right] f(x) + \eta'(1)
\end{aligned}
\tag{6.19}
$$

$$
\begin{aligned}
\Psi_2(x) = {} & \left( \langle \phi^2(t) \rangle - \langle \phi(t) \rangle^2 \right) f(x)\, \xi^2\left( \frac{1}{f(x)} \right) + f(x)\, \eta^2\left( \frac{1}{f(x)} \right) \\
& + 2(\langle t\phi(t) \rangle - \langle \phi(t) \rangle) f(x)\, \xi\left( \frac{1}{f(x)} \right) \eta\left( \frac{1}{f(x)} \right),
\end{aligned}
\tag{6.20}
$$

$$
\Psi_3(x) = (\langle t\phi(t) \rangle - \langle \phi(t) \rangle) \sqrt{f(x)}\, \xi\left( \frac{1}{f(x)} \right) + \sqrt{f(x)}\, \eta\left( \frac{1}{f(x)} \right),
\tag{6.21}
$$

and also

$$
\Psi_4(x) = \frac{\sqrt{f(x)}}{F(x)} \int_0^x \left( 1 - \frac{F(y)\, f'(y)}{f^2(y)} \right) \Psi_1(y)\, \mathrm{d}y
\tag{6.22}
$$

when the alternative density has a continuous derivative $f'(x)$ on $(0,1)$.

**Assertion 6.5.1**  Consider the observations under the fixed alternative $F \sim f$ with $f$ positive and continuous on $[0, 1]$ and continuously differentiable on $(0, 1)$ with the derivative $f'$ bounded. If $U_\phi$ is a statistic from the set $\{R_\phi, \tilde{R}_\phi, S_\phi, \tilde{S}_\phi, T_\phi, \tilde{T}_\phi\}$, and $\phi \in \mathbf{\Phi}_2$ satisfies the stronger version of (6.5) with $\beta < 1/2$, then

$$\frac{1}{\sqrt{n}}(U_\phi - n\mu_\phi(f)) \xrightarrow{\mathcal{D}} N(0, \sigma_\phi^2(f)) \quad \text{as} \quad n \to \infty \tag{6.23}$$

where $\mu_\phi(f)$ is given by (6.12) and

$$\sigma_\phi^2(f) = \int_0^1 \Psi_2(x)\, \mathrm{d}x - 2 \int_0^1 \Psi_3(x)\, \Psi_4(x)\, \mathrm{d}x + \int_0^1 \Psi_4^2(x)\, \mathrm{d}x \tag{6.24}$$

for $\Psi_2(x)$, $\Psi_3(x)$ and $\Psi_4(x)$ defined by (6.20)–(6.22). $\qquad \square$

**Remark 6.5.1**  Under the hypothesis $F_0 \sim f_0 = 1$ both Assertions 6.4.1 and 6.5.1 deal with the same statistical model. Therefore, if $f = f_0$, the asymptotic parameters $(\mu_\phi, \sigma_\phi^2)$ from (6.17) and $(\mu_\phi(f_0), \sigma_\phi^2(f_0))$ from (6.12) and (6.24) must be the same, that is, the equalities

$$\mu_\phi(f_0) = \langle \phi \rangle \quad \text{and} \quad \sigma_\phi^2(f_0) = \langle \phi^2 \rangle - \langle \phi \rangle^2 - (\langle t\phi(t) \rangle - \langle \phi \rangle)^2$$

must hold. The first equality is clear from (6.13). For $f = f_0$ we get from (9.62) by partial integration

$$\Psi_1(y) = \langle t\phi'(t) \rangle = \langle t\phi(t) \rangle - \langle \phi \rangle \quad \text{for all } y \in (0, 1).$$

Thus, by (6.22), $\Psi_4(x)$ is under the hypothesis constant, equal to $\langle t\phi(t) \rangle - \langle \phi \rangle$. Similarly, by (6.20), (6.21) and Assertion 5.3.1, $\Psi_2(x) = \langle \phi^2 \rangle - \langle \phi \rangle^2$ and $\Psi_3(x) = \Psi_4(x)$. Hence (6.24) implies the desired result

$$\sigma_\phi^2(f_0) = \Psi_2(x) - 2\Psi_4^2(x) + \Psi_4^2(x) = \sigma_\phi^2.$$

**Remark 6.5.2**  The expressions $\mu_\phi$, $\sigma_\phi^2$ are well defined by (6.17) for every continuous function $\phi : (0, \infty) \mapsto \mathbb{R}$ satisfying the condition (6.5) with $\beta < 1/2$. If this condition holds for some function $\psi : (0, \infty) \mapsto \mathbb{R}$, then it holds also for all linear transformations $\phi(t) = a\psi(t) + b(t - 1) + c$ and

$$\mu_\phi = a\mu_\psi + c, \quad \sigma_\phi^2 = a^2 \sigma_\psi^2. \tag{6.25}$$

Let us now consider a fixed alternative $F \sim f$ with the density continuously differentiable on $(0, 1)$. Then, using expression (9.52) for $\mu_\phi(f)$, and (9.54)–(9.56) for $s_i^2(f)$, the formulas

$$\mu_\phi(f) = \int_0^1 f(x) \left\langle \phi\left(\frac{t}{f(x)}\right) \right\rangle \mathrm{d}x \quad \text{and} \quad \sigma_\phi^2(f) = s_1^2(f) + s_2^2(f) + s_3^2(f) \tag{6.26}$$

define $\mu_\phi(f)$ and $\sigma_\phi^2(f)$ for all continuously differentiable functions $\phi : (0, \infty) \mapsto \mathbb{R}$ such that both $\phi(t)$ and $\tilde{\phi}(t) = t\phi'(t)$ satisfy (6.5) with $\beta < 1/2$. If $\psi$ is one of the functions satisfying all these conditions then all linear transformations $\phi(t) = a\psi(t) + b(t - 1) + c$ satisfy these conditions too and

$$\mu_\phi(f) = a\mu_\psi(f) + c, \quad \sigma_\phi^2(f) = a^2 \sigma_\psi^2(f). \tag{6.27}$$

Formulas (6.25) and (6.27) are verifiable from the definitions mentioned in this remark and are useful for the evaluation of asymptotic means and variances.

**Remark 6.5.3**  We observe that the asymptotic results of Assertions 6.3.1, 6.4.1 and 6.5.1 are in each case for a fixed $\phi$ the same for *any* statistic $U_\phi$ from the class of statistics considered, confirming the asymptotic equivalence of these statistics.

# 7 Asymptotic properties of power spacings statistics

## 7.1 Power spacing statistics

In the rest of this chapter we deal with and frequently refer to the statistics generated by the power functions $\phi = \tilde{\phi}_\alpha$ introduced in (3.26), (3.27) as nonnegative linear transforms of the simpler functions $\phi = \phi_\alpha$ defined by (3.14), (3.15). In order to simplify the notations, we interchange the symbols $\phi_\alpha \longleftrightarrow \tilde{\phi}_\alpha$, i.e. we use the functions defined by

$$\phi_\alpha(t) = \frac{t^\alpha - \alpha(t-1) - 1}{\alpha(\alpha-1)} \quad \text{if } \alpha \notin \{0,1\}, \tag{7.1}$$

and otherwise by the corresponding limits

$$\phi_1(t) = t \ln t - t + 1 \quad \text{and} \quad \phi_0(t) = -\ln t + t - 1, \tag{7.2}$$

and their simpler alternatives

$$\tilde{\phi}_\alpha(t) = \frac{t^\alpha - 1}{\alpha(\alpha-1)} \quad \text{for} \quad \alpha \notin \{0,1\}, \quad \tilde{\phi}_1(t) = t \ln t, \quad \tilde{\phi}_0(t) = -\ln t. \tag{7.3}$$

The rest of this chapter pays special attention to the subclass

$$\mathcal{U}_\alpha = \{R_{\phi_\alpha}, \tilde{R}_{\phi_\alpha}, S_{\phi_\alpha}, \tilde{S}_{\phi_\alpha}, T_{\phi_\alpha}, \tilde{T}_{\phi_\alpha}\}, \quad \alpha \in \mathbb{R} \tag{7.4}$$

of the spacings-based structural disparity statistics studied in the previous section which are generated by the power functions $\phi = \phi_\alpha : (0,\infty) \mapsto \mathbb{R}$ defined for all powers $\alpha \in \mathbb{R}$ in (7.1) and (7.2). It is easy to verify that these functions belong to the subset $\mathbf{\Phi}_2$, that is, they satisfy the functional equation (5.39) with

$$\xi(t) = \xi_\alpha(t) = t^\alpha \quad \text{and} \quad \eta(t) = \eta_\alpha(t) = \begin{cases} \frac{t^\alpha - t}{\alpha - 1} & \text{if} \quad \alpha \neq 1 \\ \lim_{\alpha \to 1} \frac{t^\alpha - t}{\alpha - 1} = t \ln t & \text{if} \quad \alpha = 1 \end{cases} \tag{7.5}$$

In other words, if $\alpha \in \mathbb{R}$ then

$$\phi_\alpha(st) = s^\alpha \phi_\alpha(t) + \phi_\alpha(s) + (t-1) \cdot \begin{cases} \frac{s^\alpha - s}{\alpha - 1} & \text{if} \quad \alpha \neq 1 \\ s \ln s & \text{if} \quad \alpha = 1 \end{cases} \tag{7.6}$$

for all $s, t > 0$.

It is also easy to verify that the functions $\phi_\alpha$, $\alpha \in \mathbb{R}$ are convex, belong to $\mathbf{\Phi}_{\mathrm{div}}$ and define $\phi_\alpha$-divergences (or briefly, $\alpha$-divergences). Referring to definitions (3.1) and (3.2), we introduce the following simplified notation for these $\phi_\alpha$-divergences:

$$D_\alpha(\boldsymbol{p}, \boldsymbol{p}_0) = D_{\phi_\alpha}(\boldsymbol{p}, \boldsymbol{p}_0) = \frac{1}{\alpha(\alpha-1)} \left( \sum_{j=1}^k p_j^\alpha p_{0j}^{1-\alpha} - 1 \right) = D_{1-\alpha}(\boldsymbol{p}_0, \boldsymbol{p}) \quad \text{(cf. (3.4))}$$

if $\alpha \notin \{0,1\}$, and

$$D_1(\boldsymbol{p}, \boldsymbol{p}_0) = D_{\phi_1}(\boldsymbol{p}, \boldsymbol{p}_0) = \sum_{j=1}^k p_j \ln \frac{p_j}{p_{0j}} = D_0(\boldsymbol{p}_0, \boldsymbol{p})$$

otherwise. Similarly (cf. (3.2)),

$$D_0(F_0, F) = D_{\phi_0}(F_0, F) = \int_0^1 f \ln \frac{f}{f_0} \, dx = \int_0^1 f(x) \ln f(x) \, dx, \tag{7.7}$$

$$D_1(F_0, F) = D_{\phi_1}(F_0, F) = \int_0^1 f_0 \ln \frac{f_0}{f} \, dx = -\int_0^1 \ln f(x) \, dx, \tag{7.8}$$

$$D_\alpha(F_0, F) = D_{\phi_\alpha}(F_0, F) = \frac{1}{\alpha(\alpha-1)} \left( \int_0^1 f \left( \frac{f_0}{f} \right)^\alpha dx - 1 \right)$$

$$= \frac{1}{\alpha(\alpha-1)} \left( \int_0^1 f(x)^{1-\alpha} dx - 1 \right) \quad \text{if} \quad \alpha \notin \{0,1\}. \tag{7.9}$$

Similar to the corresponding $\phi_\alpha$-divergences themselves, the $\phi_\alpha$-divergence statistics $T_{\phi_\alpha}$, $\tilde{T}_{\phi_\alpha}$ and $S_{\phi_\alpha}$ are not altered if the nonnegative convex functions $\phi_\alpha \in \mathbf{\Phi}_2$ are replaced by the convex functions $\phi_\alpha(t)$ from $\mathbf{\Phi}_2$ given by (7.3).

For illustration and references later we present formulas for selected statistics from the class (7.4). In the first set are our true divergence statistics

$$T_{\phi_\alpha} = T_{\phi_\alpha,n} = \frac{1}{\alpha(\alpha-1)} \left[ n^\alpha \left( \sum_{j=1}^{n-1} (Y_j - Y_{j-1})^\alpha + (1 - Y_{n-1})^\alpha \right) - n \right] \tag{7.10}$$

$$T_{\phi_1} = T_{\phi_1,n} = \sum_{j=1}^{n-1} n(Y_j - Y_{j-1}) \ln \left[ n(Y_j - Y_{j-1}) \right] + n(1 - Y_{n-1}) \ln \left[ n(1 - Y_{n-1}) \right] \tag{7.11}$$

$$T_{\phi_0} = T_{\phi_0,n} = - \sum_{j=1}^{n-1} \ln \left[ n(Y_j - Y_{j-1}) \right] - \ln \left[ (1 - Y_{n-1}) \right] \tag{7.12}$$

(cf. (5.32)). In the second set are the modified divergence statistics

$$S_{\phi_\alpha} = S_{\phi_\alpha,n} = \frac{1}{\alpha(\alpha-1)} \left[ (n+1)^\alpha \sum_{j=1}^{n+1} (Y_j - Y_{j-1})^\alpha - n - 1 \right] \tag{7.13}$$

$$S_{\phi_1} = S_{\phi_1,n} = \sum_{j=1}^{n+1} (n+1)(Y_j - Y_{j-1}) \ln \left[ n(Y_j - Y_{j-1}) \right] \tag{7.14}$$

$$S_{\phi_0} = S_{\phi_0,n} = - \sum_{j=1}^{n+1} \ln \left[ (n+1)(Y_j - Y_{j-1}) \right] \tag{7.15}$$

(cf. (5.31)) extensively used in the literature (cf. Jammalamadaka et al. (1986), (1989), Misra and van der Meulen (2001), Jiménez and Shao (2009) and others cited there). In the third set are again the true divergence statistics

$$\tilde{T}_{\phi_\alpha} = \tilde{T}_{\phi_\alpha,n} = \frac{1}{\alpha(\alpha-1)} \left[ n^\alpha \left( \sum_{j=2}^{n} (Y_j - Y_{j-1})^\alpha + (Y_1 + 1 - Y_n)^\alpha \right) - n \right] \tag{7.16}$$

$$\tilde{T}_{\phi_1} = \tilde{T}_{\phi_1,n} = \sum_{j=2}^{n} n(Y_j - Y_{j-1}) \ln \left[ n(Y_j - Y_{j-1}) \right] + n(Y_1 + 1 - Y_n) \ln \left[ n(Y_1 + 1 - Y_n) \right] \tag{7.17}$$

$$\tilde{T}_{\phi_0} = \tilde{T}_{\phi_0,n} = - \sum_{j=2}^{n} \ln \left[ n(Y_j - Y_{j-1}) \right] - \ln \left[ n(Y_1 + 1 - Y_n) \right] \tag{7.18}$$

(cf. (5.35)) slightly different from those used by Hall (1986), Morales et al. (2003), Vajda and van der Meulen (2006), Vajda (2007) and others cited there (they differ by the normalizing constant $n$ instead of $n+1$).

**Example 7.1.1**   For $\alpha = 2$ we obtain the statistic

$$S_{\phi_2} = \frac{1}{2} \left[ (n+1)^2 \sum_{j=1}^{n+1} (Y_j - Y_{j-1})^2 - (n+1) \right] = \frac{n+1}{2} \left[ (n+1)\mathcal{G} - 1 \right] \tag{7.19}$$

where $\mathcal{G}$ is the Greenwood statistic of Example 5.1.1.

Since the general asymptotic theory of the statistics $U_\alpha \in \mathcal{U}_\alpha$ specified by (7.4) is covered by Assertion 5.3.2 and Assertions 6.3.1–6.5.1, the assertions that follow in this section are basically their corollaries.

However, they bring explicit formulas and additional important new results, the proofs of which are not trivial. These proofs are partly based on a continuity theory for the asymptotic parameters

$$\mu_\alpha(f) = \mu_{\phi_\alpha}(f), \ \sigma_\alpha^2(f) = \sigma_{\phi_\alpha}^2(f), \ \mu_\alpha = \mu_{\phi_\alpha}, \ \sigma_\alpha^2 = \sigma_{\phi_\alpha}^2 \text{ and } m_\alpha(\ell) = m_{\phi_\alpha}(\ell), \quad (7.20)$$

defined by (6.26), (6.17) and (6.18), as functions of the structural parameter $\alpha \in \mathbb{R}$. Such a theory enables us to avoid a direct calculation of the asymptotic parameters at some $\alpha_0 \in \mathbb{R}$, if these calculations are tedious and the asymptotic parameters are known at the neighboring parameters $\alpha$. This theory is summarized in Assertion 7.1.2 below using Assertion 7.1.1. In Assertion 7.1.2 we take the representations (6.12) and (6.24) for $\mu_{\phi_\alpha}(f)$ and $\sigma_{\phi_\alpha}^2(f)$ rather than (6.26).

**Assertion 7.1.1** Let $g(y)$ be a continuous positive function on a compact interval $[a, b] \subset \mathbb{R}$ and $\Phi(u, v)$ a continuous function of variables $u, v \in \mathbb{R}$. Furthermore let, for all $\alpha$ from an interval $(c, d) \subset \mathbb{R}$, $\psi_\alpha : (0, \infty) \mapsto \mathbb{R}$ be convex or concave functions differentiable at some point $t_* \in (0, \infty)$. If the values $\psi_\alpha(t)$, $t \in (0, \infty)$, and the derivatives $\psi_\alpha'(t_*)$ depend continuously on $\alpha \in (c, d)$, then for every $\alpha_0 \in (c, d)$

$$\lim_{\alpha \to \alpha_0} \int_a^b \Phi(g, \psi_\alpha(g)) \, dy = \int_a^b \Phi(g, \psi_{\alpha_0}(g)) \, dy. \quad (7.21)$$

$\square$

**Assertion 7.1.2** The asymptotic parameters $\mu_\alpha$, $\sigma_\alpha^2$ and $m_\alpha(\ell)$, specified by (7.20), (6.17) and (6.18), are continuous in the variable $\alpha \in (-1/2, \infty)$. If the density $f$ satisfies the assumptions of Assertion 6.3.1, then the asymptotic mean $\mu_\alpha(f)$ specified by (7.20) and (6.12) is continuous in the variable $\alpha \in (-1, \infty)$. If $f$ satisfies the stronger assumptions of Assertion 6.5.1, then the asymptotic variance $\sigma_\alpha^2(f)$ specified by (7.20) and (6.24) is continuous in the variable $\alpha \in (-1/2, \infty)$. $\square$

## 7.2 Consistency

In the assertion below and in the rest of the chapter, we use the gamma function of the variable $\alpha \in \mathbb{R}$ and the Euler constant,

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} \, dt \quad \text{and} \quad \gamma = 0.577 \ldots. \quad (7.22)$$

**Assertion 7.2.1** Consider the observations under the fixed alternative $F \sim f$ assumed in Assertion 6.3.1 and denote by $U_\alpha$ any statistic from the class $\mathcal{U}_\alpha$ of (7.4). If $\alpha > -1$, then

$$\frac{U_\alpha}{n} \xrightarrow{p} \mu_\alpha(f) \quad \text{as } n \to \infty \quad (7.23)$$

for

$$\mu_\alpha(f) = D_\alpha(F_0, F) \, \Gamma(\alpha + 1) + \mu_\alpha, \quad (7.24)$$

where

$$\mu_0 = \gamma, \quad \mu_1 = 1 - \gamma, \quad \text{and} \quad \mu_\alpha = \frac{\Gamma(\alpha + 1) - \Gamma(1)}{\alpha(\alpha - 1)} \quad \text{for } \alpha \notin \{0, 1\}, \quad (7.25)$$

and $D_\alpha(F_0, F)$ are the $\phi_\alpha$-divergences (7.7)-(7.9). The $\phi_\alpha$-divergences are zero if and only if $F = F_0$, so that under the hypothesis $\mathcal{H}_0 : F = F_0$

$$\mu_\alpha(f_0) = \mu_\alpha, \quad \alpha \in \mathbb{R}. \quad (7.26)$$

Both parameters $\mu_\alpha$ and $\mu_\alpha(f)$ are continuous in the variable $\alpha \in (-1, \infty)$ and satisfy the inequality $\mu_\alpha(f) \geq \mu_\alpha$, which is strict unless $F = F_0$. $\square$

Since $\Gamma(\alpha + 1) = \alpha(\alpha - 1)\Gamma(\alpha - 1)$, (7.25) and (7.24) can be replaced for $\alpha \notin \{0, 1\}$ by

$$\mu_\alpha = \Gamma(\alpha - 1) - \frac{1}{\alpha(\alpha - 1)} \quad \text{and} \quad \mu_\alpha(f) = \Gamma(\alpha - 1) \int_0^1 f^{1-\alpha} \, dx - \frac{1}{\alpha(\alpha - 1)}. \quad (7.27)$$

Assertion 7.2.1 can be illustrated by Table 7.2.1 , in which actual values of the parameters $\mu_\alpha$ and $\mu_\alpha(f)$ are presented for selected parameters $\alpha$. In this table, $f$ denotes any density considered in Assertion 6.3.1, and the expressions for $D_\alpha(F_0, F)$, $H(F_0, F)$, and $\chi^2(F_0, F)$ can be easily discerned from those used in Examples 3.1.1 and 3.1.3, thereby replacing $P$ and $P_0$ by $F_0$ and $F$ and sums by integrals.

**Table 7.2.1** Values of $\mu_\alpha$ and $\mu_\alpha(f)$ for selected $\alpha > -1$.

| $\alpha$ | $\mu_\alpha$ | $\mu_\alpha(f)$ |
|---|---|---|
| $-\frac{1}{2}$ | $\frac{4}{3}(\sqrt{\pi} - 1) \doteq 1.030$ | $\sqrt{\pi}\,D_{-1/2}(F_0, F) + \mu_{-1/2} = \frac{4\sqrt{\pi}}{3}\int_0^1 f^{3/2}\,\mathrm{d}x - \frac{4}{3}$ |
| $0$ | $\gamma \doteq 0.577$ | $D_0(F_0, F) + \mu_0 = \int_0^1 f\ln f\,\mathrm{d}x + \gamma$ |
| $\frac{1}{2}$ | $4 - 2\sqrt{\pi} \doteq 0.455$ | $2\sqrt{\pi}\,H(F_0, F) + \mu_{1/2} = 4 - 2\sqrt{\pi}\int_0^1 \sqrt{f}\,\mathrm{d}x$ |
| $1$ | $1 - \gamma \doteq 0.423$ | $D_1(F_0, F) + \mu_1 = 1 - \gamma - \int_0^1 \ln f\,\mathrm{d}x$ |
| $\frac{3}{2}$ | $\sqrt{\pi} - \frac{4}{3} \doteq 0.439$ | $\frac{3\sqrt{\pi}}{4}D_{3/2}(F_0, F) + \mu_{3/2} = \sqrt{\pi}\int_0^1 \frac{\mathrm{d}x}{\sqrt{f}} - \frac{4}{3}$ |
| $2$ | $\frac{1}{2} = 0.500$ | $\chi^2(F_0, F) + \mu_2 = \int_0^1 \frac{\mathrm{d}x}{f} - \frac{1}{2}$ |
| $\frac{5}{2}$ | $\frac{\sqrt{\pi}}{2} - \frac{4}{15} \doteq 0.620$ | $\frac{15\sqrt{\pi}}{8}D_{5/2}(F_0, F) + \mu_{5/2} = \frac{\sqrt{\pi}}{2}\int_0^1 \frac{\mathrm{d}x}{f^{3/2}} - \frac{4}{15}$ |
| $3$ | $\frac{5}{6} \doteq 0.833$ | $6D_3(F_0, F) + \mu_3 = \int_0^1 \frac{\mathrm{d}x}{f^2} - \frac{1}{6}$ |

## 7.3 Asymptotic normality under local alternatives

**Assertion 7.3.1** Consider the observations under the local alternatives (6.1) with the limit function $\ell(x)$ introduced in (6.2), and denote by $U_\alpha$ any statistic from the class $\mathcal{U}_\alpha$ of (7.4). If $\alpha > -1/2$, then

$$\frac{1}{\sqrt{n}}(U_\alpha - n\mu_\alpha) \xrightarrow{\mathcal{D}} N(m_\alpha(\ell), \sigma_\alpha^2) \qquad \text{as } n \to \infty \tag{7.28}$$

where the parameters $\mu_\alpha$, $m_\alpha(\ell)$, and $\sigma_\alpha^2$ are continuous in the variable $\alpha \in (-1/2, \infty)$ , and are given by (7.25) and the formulas

$$m_\alpha(\ell) = \frac{\|\ell\|^2}{2}\,\Gamma(\alpha + 1) \tag{7.29}$$

$$\sigma_\alpha^2 = \frac{\Gamma(2\alpha + 1) - (\alpha^2 + 1)\,\Gamma^2(\alpha + 1)}{\alpha^2(\alpha - 1)^2} \qquad \text{for } \alpha \notin \{0, 1\} \tag{7.30}$$

and

$$\sigma_0^2 = \frac{\pi^2}{6} - 1, \qquad \sigma_1^2 = \frac{\pi^2}{3} - 3. \tag{7.31}$$

$\square$

Assertion 7.3.1 provides the possibility to compute and compare asymptotic relative efficiencies of tests of the hypothesis $\mathcal{H}_0 : F_0 \sim f_0$ based on the statistics $U_\alpha \in \mathcal{U}_\alpha$, $\alpha > -1/2$, for various values of $\alpha$. The Pitman asymptotic relative efficiency (ARE) of one test relative to another is defined as the limit of the inverse ratio of sample sizes required to obtain the same limiting power at the sequence of alternatives converging to the null hypothesis. If we define the "efficacies" of the statistics $U_\alpha \in \mathcal{U}_\alpha$ of Assertion 7.3.1 by

$$\mathrm{eff}(U_\alpha) = \frac{\Gamma^2(\alpha + 1)}{\sigma_\alpha^2} = \frac{(m_\alpha(\ell))^2}{\sigma_\alpha^2}\left(\frac{2}{\|\ell\|^2}\right)^2 \qquad \text{for } \|\ell\|^2 \neq 0$$

then under the assumptions of Assertion 7.3.1 we get in accordance with Section 4 in Del Pino (1979)

$$\mathrm{ARE}(U_{\alpha_1}, U_{\alpha_2}) = \frac{\mathrm{eff}(U_{\alpha_1})}{\mathrm{eff}(U_{\alpha_2})}$$

where $U_{\alpha_1}$ and $U_{\alpha_2}$ are arbitrary statistics from $\mathcal{U}_{\alpha_1}$ and $\mathcal{U}_{\alpha_2}$. Notice that arbitrary statistics $U_\alpha$ from the set $\mathcal{U}_\alpha$, $\alpha$ fixed, all have the same efficacy (cf. also Remark 6.5.3). In Table 7.3.1 we present the parameters $m_\alpha(\ell)$, $\sigma_\alpha^2$ and $\Gamma^2(\alpha+1)/\sigma_\alpha^2$ for selected values of $\alpha > -1/2$. This table indicates that the statistics $U_2 \in \{R_{\phi_2}, \tilde{R}_{\phi_2}, S_{\phi_2}, \tilde{S}_{\phi_2}, T_{\phi_2}, \tilde{T}_{\phi_2}\}$ are most asymptotically efficient in the Pitman sense among all statistics $U_\alpha$, $\alpha > -1/2$. This extends the result on p. 1457 in Rao and Kuo (1984) about the asymptotic efficiency of the Greenwood statistic $\mathcal{G} = (2S_{\phi_2} + n + 1)/(n+1)^2$ (cf. Example 5.1.1 (formula (5.4)), Example 7.1.1 (formula (7.19)), and formula (7.46) below).

**Table 7.3.1** The asymptotic parameters $m_\alpha(\ell)$, $\sigma_\alpha^2$ and $\mathrm{eff}(U_\alpha)$

for selected statistics $U_\alpha$ of Assertion 7.3.1.

| $\alpha$ | $m_\alpha(\ell)$ | $\sigma_\alpha^2$ | $\mathrm{eff}(U_\alpha)$ |
|---|---|---|---|
| 0 | $\frac{\|\ell\|^2}{2}$ | $\frac{\pi^2}{6} - 1 \doteq 0.645$ | 1.550 |
| $\frac{1}{2}$ | $\|\ell\|^2 \frac{\sqrt{\pi}}{4} \doteq \frac{\|\ell\|^2}{2} \times 0.886$ | $16 - 5\pi \doteq 0.292$ | 2.690 |
| 1 | $\frac{\|\ell\|^2}{2}$ | $\frac{\pi^2}{3} - 3 \doteq 0.290$ | 3.448 |
| $\frac{3}{2}$ | $\|\ell\|^2 \frac{3\sqrt{\pi}}{8} \doteq 1.329$ | $\frac{32}{3} - \frac{13\pi}{4} \doteq 0.457$ | 3.871 |
| 2 | $\|\ell\|^2 = \frac{\|\ell\|^2}{2} \times 2$ | 1 | 4.000 |
| $\frac{5}{2}$ | $\|\ell\|^2 \frac{15\sqrt{\pi}}{16} \doteq \frac{\|\ell\|^2}{2} \times 3.323$ | $\frac{128}{15} - \frac{29\pi}{16} \doteq 2.839$ | 3.890 |
| 3 | $\|\ell\|^2 \, 3 = \frac{\|\ell\|^2}{2} \times 6$ | 10 | 3.600 |

The general form of the asymptotic normality (7.28), as well as the continuity of the parameters $\mu_\alpha$, $m_\alpha(\ell)$ and $\sigma_\alpha^2$ in $\alpha \in (-1/2, \infty)$ established in Assertion 7.3.1 appear to be new results. The special result for $\alpha = 0$ also seems to be new. The particular result for $\alpha \in (-1/2, \infty) - \{0, 1\}$ and $U_\alpha = S_{\phi_\alpha}$ follows from the asymptotic normality obtained for the statistics

$$\sum_{j=1}^{n+1} ((n+1)(Y_j - Y_{j-1}))^\alpha = \alpha(\alpha-1) S_{\phi_\alpha} + n + 1 \tag{7.32}$$

(cf. (7.47) below) by Del Pino, see p. 1062 in Del Pino (1979). The particular result for $\alpha = 1$ and the statistic $U_1 = S_{\phi_1}$ with $\mu_1$ and $\sigma_1^2$ given in Tables 7.2.1 and 7.3.1 was obtained previously by Misra and van der Meulen (2001), who however considered $m$-spacings for arbitrary $m \geq 1$. They compared also the efficiency of the test statistics for $\alpha = 0$, $\alpha = 1$ and $\alpha = 2$ with a similar conclusion as in Table 7.3.1.

## 7.4 Asymptotic normality under fixed alternatives

In this subsection we study the asymptotic distributions of the spacings-type power divergence statistics $U_\alpha$ from the sets $\mathcal{U}_\alpha = \{R_{\phi_\alpha}, \tilde{R}_{\phi_\alpha}, S_{\phi_\alpha}, \tilde{S}_{\phi_\alpha}, T_{\phi_\alpha}, \tilde{T}_{\phi_\alpha}\}$ for $\alpha > -1/2$ under the assumption that the observations are distributed by a fixed alternative $F \sim f$ satisfying the assumptions of Assertion 6.5.1. If $\alpha > -1/2$ then $\phi_\alpha$ satisfies the assumption of Assertion 6.5.1 too. Therefore this theorem implies that

$$\frac{1}{\sqrt{n}}(U_\alpha - n\mu_\alpha(f)) \xrightarrow{\mathcal{D}} N(0, \sigma_\alpha^2(f)) \quad \text{for } n \to \infty \tag{7.33}$$

where the asymptotic parameters $\mu_\alpha(f)$, $\sigma_\alpha^2(f)$ are given by (7.20). Similarly as in the previous section, we are interested in explicit formulas for these parameters. By Assertion 6.5.1, the asymptotic mean is

for all $\alpha \in \mathbb{R}$ given by the explicit formula (7.24) presented in Assertion 7.2.1. The only problem which remains is the formula for $\sigma_\alpha^2(f)$, $\alpha \in \mathbb{R}$.

The functions $\psi_\alpha(t) = t^\alpha$ with $\alpha > -1/2$ satisfy all assumptions of Remark 6.5.2 so that we can consider the quantities

$$\tau_\alpha^2(f) := \sigma_{\psi_\alpha}^2(f), \quad \alpha \in (-1/2, \infty)$$

obtained from the formula introduced there. By (6.27),

$$\sigma_\alpha^2(f) = \frac{\tau_\alpha^2(f)}{\alpha^2(\alpha - 1)^2} \quad \text{for } \alpha \in (-1/2, \infty) - \{0, 1\}. \tag{7.34}$$

One can find on p. 521 of Hall (1984) an expression for $\tau_\alpha^2(f)$ for all $\alpha \in (-1/2, \infty) - \{0, 1\}$, which for the case $m = 1$ can be given the form

$$\tau_\alpha^2(f) = \alpha^2(\alpha - 1)^2 \left( \sigma_\alpha^2 \int_0^1 f^{1-2\alpha} \mathrm{d}x + \Gamma^2(\alpha + 1) \Delta_\alpha(F_0, F) \right) \tag{7.35}$$

where $\sigma_\alpha^2$ is defined by formula (7.30) and

$$\Delta_\alpha(F_0, F) = \frac{1}{\alpha^2} \int_0^1 \left( \frac{1}{(f(x))^\alpha} - \frac{1}{F(x)} \int_0^x (f(y))^{1-\alpha} \mathrm{d}y \right)^2 f(x) \, \mathrm{d}x \quad \text{for } \alpha \in \mathbb{R} - \{0\}. \tag{7.36}$$

Since Hall (1984) gave no hint about the derivation of his formula, let us mention that (7.35) is obtained if one substitutes $\psi_\alpha$ for $\phi$ in the expressions (9.54)– (9.56) below for $s_j^2(f)$, $j \in \{1, 2, 3\}$, given in the proof of Assertion 6.5.1 ( thereby employing the expression

$$\begin{aligned}
G(x) &= \alpha E(Z^\alpha) \int_0^x \left( 1 - \frac{Ff'}{f^2} \right) \frac{1}{f^{\alpha-1}} \, \mathrm{d}y \\
&= \Gamma(\alpha + 1) \left( (\alpha - 1) \int_0^x (f(y))^{1-\alpha} \mathrm{d}y + (f(x))^{-\alpha} F(x) \right)
\end{aligned}$$

for $G(x)$ of (9.53) when $\phi$ is replaced by $\psi_\alpha$ ), and then forms the sum $s_1^2(f) + s_2^2(f) + s_3^2(f)$. By (7.34) and (7.35),

$$\sigma_\alpha^2(f) = \sigma_\alpha^2 \int_0^1 f^{1-2\alpha} \mathrm{d}x + \Gamma^2(\alpha + 1) \Delta_\alpha(F_0, F), \quad \alpha \in (-1/2, \infty) - \{0, 1\}. \tag{7.37}$$

The final, intuitively appealing, form of the asymptotic variance

$$\sigma_\alpha^2(f) = (1 + 2\alpha(2\alpha - 1) D_{2\alpha}(F_0, F)) \sigma_\alpha^2 + \Gamma^2(\alpha + 1) \Delta_\alpha(F_0, F) \tag{7.38}$$

(with $\sigma_\alpha^2(f_0) = \sigma_\alpha^2$ given in (7.30)), follows for $\alpha \in (-1/2, \infty) - \{0, 1\}$ by taking into account the formula for $D_{2\alpha}(F_0, F)$ obtained from (7.9). The peculiar expression $\Delta_\alpha(F_0, F)$ figuring in (7.36) and (7.38) can be better understood if we take into account the following assertion, after which we extend (7.38) to include also the values $\alpha \in \{0, 1\}$.

**Assertion 7.4.1** If the fixed alternative $F \sim f$ satisfies the assumptions of Assertion 6.5.1, then the class $\{\Delta_\alpha(F_0, F) : \alpha \in \mathbb{R} - \{0\}\}$ consists of the variances

$$\begin{aligned}
\Delta_\alpha(F_0, F) &= \int_0^1 \left( \frac{f^{-\alpha}}{\alpha} - \int_0^1 \frac{f^{-\alpha}}{\alpha} f \, \mathrm{d}y \right)^2 f \, \mathrm{d}x \\
&= \int_0^1 \left( \frac{f^{-\alpha}}{\alpha} \right)^2 f \, \mathrm{d}x - \left( \int_0^1 \frac{f^{-\alpha}}{\alpha} f \, \mathrm{d}x \right)^2
\end{aligned} \tag{7.39}$$

of the functions $f^{-\alpha}(X)/\alpha$ of the random argument $X$ distributed by $F$. This class is continuously extended to all $\alpha \in \mathbb{R}$ by introducing the variance

$$\begin{aligned}
\Delta_0(F_0, F) &= \int_0^1 \left( \ln f - \int_0^1 (\ln f) f \, \mathrm{d}y \right)^2 f \, \mathrm{d}x \\
&= \int_0^1 f \ln^2 f \, \mathrm{d}x - \left( \int_0^1 f \ln f \, \mathrm{d}x \right)^2
\end{aligned} \tag{7.40}$$

of the function $\ln f(X)$ of the random argument $X$ introduced above. All $\Delta_\alpha(F_0, F)$, $\alpha \in \mathbb{R}$, are nonnegative measures of divergence of $F_0$ and $F$, reflexive in the sense that $\Delta_\alpha(F_0, F) = 0$ if and only if $F = F_0$.

$\square$

We are now in a position to formulate the general results obtained in this chapter regarding the asymptotic normality of spacings-type power divergence statistics $U_\alpha$ from the sets $\mathcal{U}_\alpha = \{R_{\phi_\alpha}, \tilde{R}_{\phi_\alpha}, S_{\phi_\alpha}, \tilde{S}_{\phi_\alpha}, T_{\phi_\alpha}, \tilde{T}_{\phi_\alpha}\}$ for $\alpha > -1/2$ under the assumption of the fixed alternative, thereby specifying the parameters $\mu_\alpha(f)$ and $\sigma_\alpha^2(f)$ in (7.33) for all $\alpha > -1/2$. Inspecting once more formula (7.38), we observe that if $\alpha > -1/2$ differs from 0 and 1, then the asymptotic variance $\sigma_\alpha^2(f)$ under the alternative $f$ exceeds the asymptotic variance $\sigma_\alpha^2 = \sigma_\alpha^2(f_0)$ achieved under the hypothesis $F_0 \sim f_0$ by a linear function of $\sigma_\alpha^2$ given by

$$2\alpha(2\alpha - 1) D_{2\alpha}(F_0, F)\sigma_\alpha^2 + \Gamma^2(\alpha + 1) \Delta_\alpha(F_0, F) \tag{7.41}$$

with the coefficients $D_{2\alpha}(F_0, F)$ and $\Delta_\alpha(F_0, F)$ positive unless $F = F_0$. By using Assertion 7.1.2, we can now find the formulas for $\sigma_0^2(f)$ and $\sigma_1^2(f)$ which are missing in (7.37) by taking limits in (7.38) for $\alpha \to 0$ and $\alpha \to 1$. Since the limits $\sigma_0^2$ and $\sigma_1^2$ were already calculated in Assertion 7.2.1, and the limit $\Delta_0(F_0, F)$ is clear from Assertion 7.4.1, we obtain

$$\sigma_0^2(f) = \lim_{\alpha \to 0} \sigma_\alpha^2(f) = \sigma_0^2 + \Delta_0(F_0, F) \tag{7.42}$$

and

$$\sigma_1^2(f) = \lim_{\alpha \to 1} \sigma_\alpha^2(f) = (1 + 2D_2(F_0, F)) \sigma_1^2 + \Delta_1(F_0, F) \tag{7.43}$$

where (cf. (7.39))

$$\Delta_1(F_0, F) = \int_0^1 \frac{1}{f} \mathrm{d}x - 1. \tag{7.44}$$

Together with (7.37), (7.42) and (7.43) provide formulas for $\sigma_0^2(f)$ for all $\alpha > -1/2$. It is clear that $\sigma_0^2(f)$ and $\sigma_1^2(f)$ are of the form (7.38), so that the representation (7.38) holds for all $\alpha > -1/2$. We summarize our results as follows.

**Assertion 7.4.2** If the alternative $F \sim f$ satisfies the assumptions of Assertion 6.5.1, then the asymptotic formula of (7.33) is valid for all $\alpha > -1/2$. The asymptotic means $\mu_\alpha(f)$ are given by the explicit formulas (7.24)–(7.9). The asymptotic variances $\sigma_\alpha^2(f)$ are given by (7.38), where the explicit formulas for $D_{2\alpha}(F_0, F)$ can be found in (7.7)–(7.9), those for $\sigma_\alpha^2$ in (7.30) and (7.31), and the formulas for $\Delta_\alpha(F_0, F)$ in (7.39) and (7.40). The asymptotic means and variances are continuous in the variable $\alpha \in (-1/2, \infty)$. The asymptotic means satisfy the inequality $\mu_\alpha(f) \geq \mu_\alpha$ mentioned in Assertion 7.2.1. The asymptotic variances satisfy the inequality $\sigma_\alpha^2(f) \geq \sigma_\alpha^2$. Both inequalities become equalities if and only if $F = F_0$.

$\square$

Concrete forms of $\mu_\alpha(f)$ and $\sigma_\alpha^2(f_0) = \sigma_\alpha^2$ were illustrated in Tables 7.2.1 and 7.3.1. The next table illustrates $\sigma_\alpha^2(f)$ given by (7.38) for arbitrary $f$ satisfying the assumptions of Assertion 6.5.1 and selected values of $\alpha$. In each line of Table 7.4.1 two expressions for $\sigma_\alpha^2(f)$ are given: the first one is obtained by substituting $\alpha$ in (7.38), the second one by actually calculating $D_{2\alpha}(F_0, F)$ and $\Delta_\alpha(F_0, F)$ in each case and putting the results in a closed form. As presumed, for $f = 1$ the illustrated values reduce to $\sigma_\alpha^2$ from Table 7.3.1 .

**Table 7.4.1:** Asymptotic variances $\sigma^2_\alpha(f)$ for selected $\alpha > -1/2$.

| $\alpha$ | | $\sigma^2_\alpha(f)$ | |
|---|---|---|---|
| 0 | $\sigma^2_0 + \Delta_0(F_0, F)$ | $=$ | $\frac{\pi^2}{6} - 1 + \int_0^1 f \ln^2 f \, dx \; - \; \left( \int_0^1 f \ln f \, dx \right)^2$ |
| $\frac{1}{2}$ | $\sigma^2_{\frac{1}{2}} + \frac{\pi}{4}\Delta_{\frac{1}{2}}(F_0, F)$ | $=$ | $16 - 4\pi - \pi \left( \int_0^1 \sqrt{f} \, dx \right)^2$ |
| 1 | $[1 + \chi^2(F_0, F)]\, \sigma^2_1 + \Delta_1(F_0, F)$ | $=$ | $\int_0^1 \frac{dx}{f} \left( \frac{\pi^2}{3} - 2 \right) - 1$ |
| 2 | $[1 + 12 D_4(F_0, F)\, \sigma^2_2 + 4\Delta_2(F_0, F)]$ | $=$ | $2 \int_0^1 \frac{dx}{f^3} - \left( \int_0^1 \frac{dx}{f} \right)^2$ |
| 3 | $[1 + 30 D_6(F_0, F)]\, \sigma^2_3 + 36\Delta_3(F_0, F)$ | $=$ | $14 \int_0^1 \frac{dx}{f^5} - 4 \left( \int_0^1 \frac{dx}{f^2} \right)^2$ |

## 7.5 Discussion

The general form of the asymptotic normality (7.33) established by Assertion 7.4.2, as well as the continuity of the asymptotic means and variances $\mu_\alpha(f)$ and $\sigma^2_\alpha(f)$ in the parameter $\alpha > -1/2$ proved in Assertion 7.1.2, and the explicit formulas (7.24) and (7.38) for these parameters for general $\alpha$ seem to be new results. However, in the references cited in Subsections 5.1 and 5.2 one can find particular versions of these results for some of the statistics $U_\alpha$ from the set $\{R_{\phi_\alpha}, \tilde{R}_{\phi_\alpha}, S_{\phi_\alpha}, \tilde{S}_{\phi_\alpha}, T_{\phi_\alpha}, \tilde{T}_{\phi_\alpha}\}$ or their linear functions, and for some $\alpha > -1/2$ and some distributions $F \sim f$.

Let us start with the statistic $S_{\phi_0}$ proposed by Moran (1951), and denoted by $\mathcal{M}$ in Example 5.1.2 (equation(5.7)). The asymptotic normality (7.33) for $\alpha = 0$, $U_0 = S_{\phi_0}$ and $f = f_0 = 1$, with the parameters $\mu_0(f_0) = \mu_0$ and $\sigma^2_0(f_0) = \sigma^2_0$ given in Tables 7.2.1 and 7.3.1, was proved by Darling (1953), yielding specifically that under $\mathcal{H}_0$

$$\frac{1}{\sqrt{n}} (\mathcal{M} - n\gamma) \xrightarrow{\mathcal{D}} N\left(0, \frac{\pi^2}{6} - 1\right) \quad \text{as } n \to \infty. \tag{7.45}$$

The result of Darling was extended to all positively valued step functions $f$ on $[0, 1]$ by Cressie (1976), who also obtained $\mu_0(f)$ and $\sigma^2_0(f)$ given in Tables 7.3.1 and 7.4.1. The result of Cressie was extended by van Es (1992) to the alternatives $f$ considered in the present chapter which satisfy a Lipschitz condition on $[0, 1]$, and to all $f$ considered in this chapter by Shao and Hahn (1995). Cressie(1976) and van Es(1992) studied $S_{\phi_0}$ as the special case obtained for $m = 1$ from a more general statistic based on $m$-spacings with $m \geq 1$. Van Es extended ideas and methods developed for $m > 1$ by Vasicek (1976) and Dudewicz and van der Meulen (1981) for proving the consistency and asymptotic normality of a spacings-based estimator of entropy. The latter authors considered only $\phi(t) = -\ln t$.

Greenwood (1946) introduced the statistic

$$\mathcal{G} = \sum_{j=1}^{n+1} (Y_j - Y_{j-1})^2 = \frac{2 S_{\phi_2} + n + 1}{(n+1)^2}, \tag{7.46}$$

discussed in Examples 5.1.1 and 7.1.1. Kimball (1950) proposed the generalization

$$\sum_{j=1}^{n+1} (Y_j - Y_{j-1})^\alpha = \frac{\alpha(\alpha-1) S_{\phi_\alpha} + n + 1}{(n+1)^\alpha}, \quad \alpha > 0, \tag{7.47}$$

and Darling (1953) proved an asymptotic normality theorem equivalent to (7.33) for $U_\alpha = S_{\phi_\alpha}$, $\alpha \in (0, \infty) - \{1\}$, and $f = f_0 = 1$. Weiss (1957) extended this result of Darling to positive piecewise constant densities $f$. Hall (1984) obtained the asymptotic normality

$$\frac{1}{\sqrt{n}} \left( \tilde{U}_\alpha - n\alpha(\alpha-1)\, \mu_\alpha(f) - n \right) \xrightarrow{\mathcal{D}} N(0, \alpha^2(\alpha-1)^2 \sigma^2_\alpha(f)) \quad \text{as } n \to \infty \tag{7.48}$$

for all statistics

$$
\begin{aligned}
\tilde{U}_\alpha &= \sum_{j=2}^{n} \left( n(Y_j - Y_{j-1}) \right)^\alpha \\
&= \alpha(\alpha - 1)\, \tilde{R}_{\phi_\alpha} - \alpha n (1 - Y_n + Y_1) + n + \alpha - 1 \\
&= \alpha(\alpha - 1)\, \tilde{R}_{\phi_\alpha} + n + O_p(1)
\end{aligned}
$$

with $\alpha \in (-1/2, \infty) - \{0, 1\}$ for any $f$ considered in Assertion 7.4.2. Here $\mu_\alpha(f)$ and $\sigma_\alpha^2(f)$ are the same as in Assertion 7.4.2, with $\mu_\alpha(f)$ given by the right-hand side of (7.27) and $\sigma_\alpha^2(f)$ by (7.38), $\tilde{R}_{\phi_\alpha}$ is defined as in (5.25) with $\phi = \phi_\alpha$, and the $O_p(1)$ statement follows from the proof of Assertion 6.1.1. In fact, this result of Hall (1984) was one of the arguments used in the proof of Assertion 7.4.2.

The statistic $S_{\phi_1}$ was proposed by Misra and van der Meulen (2001), who proved the asymptotic normality (7.33) for $U_1 = S_{\phi_1}$ and any $f$ considered in Assertion 7.4.2, with the parameters $\mu_1(f)$ and $\sigma_1^2(f)$ given in Tables 7.2.1 and 7.3.1, yielding the result

$$
\frac{1}{\sqrt{n}} \left( S_{\phi_1} - n \left( 1 - \gamma - \int_0^1 \ln f \, \mathrm{d}x \right) \right) \xrightarrow{\mathcal{D}} N \left( 0, \int_0^1 \left( \frac{\pi^2}{3} - 2 \right) \frac{\mathrm{d}x}{f} - 1 \right) \tag{7.49}
$$

as $n \to \infty$. We see that the present Assertion 7.4.2 unifies and extends the results proved separately in the literature in three different situations for two particular statistics from the set (7.4). The formulas for all asymptotic parameters $\mu_\alpha(f)$ and $\sigma_\alpha^2(f)$ of the statistics $U_\alpha$ are shown to follow via the asymptotic equivalence of these statistics (cf. Assertion 6.1.1) and the continuity of these parameters in $\alpha$ (cf. Assertion 7.1.2) from Hall's formula (cf. (7.48)) for the asymptotic parameters of $\tilde{U}_\alpha$ with $\alpha \in (-1/2, \infty)$ different from 0 and 1.

# 8 PODISTAT program package

This section presents a computer program package PODISTAT for evaluation of power divergence statistics. It consists of three MATLAB based functions for evaluation of three selected families of spacings based statistics parametrized by $\alpha \in \mathbb{R}$ from the class (7.4) of Section 7. Two of them are the families of the true divergence statistics $T_{\phi_\alpha, n}$ and $\tilde{T}_{\phi_\alpha, n}$ proposed in this chapter, and the third one is one of the families of classical spacings based statistic studied in this chapter, namely $S_{\phi_\alpha, n}$. The statistics $T_{\phi_\alpha, n}$, $S_{\phi_\alpha, n}$ and $\tilde{T}_{\phi_\alpha, n}$ were explicitly defined for all $\alpha \in \mathbb{R}$ by the formulas (7.10) - (7.18). The programs are illustrated by applications to the models introduced in Section 2 for the data of Tables 1.1 and 1.2.

Input parameters of the mentioned MATLAB functions are the row vector of observed data **X**, the string definition of the hypothetical distribution function **F0** and the order **alpha** of the statistic.

**1.** Evaluation of the statistic $T_{\phi_\alpha, n}$ given by (7.10) - (7.12) and denoted as function T = Tpdt.

```
function T = Tpdt(X, F0, alpha)
% This function compute the Goodness-of-fit statistic based on
% the power-divergences.
%
% Use: Tpdt(X, F0, {alpha})
%    x      row vector; observed data
%    F0     string: hypothetical cumulative distribution distribution of vector x
%    {alpha} number: from interval (-1/2, inf) order of power-divergence
%                   (optional, default = 2)
%
% Example: Tpdt(X, 'normal_cdf(x, 0, 1)', 1.5)
eps = 1.E-4;
if (nargin<2), error('Use: Tpdt(x, F0, {alpha})'); end
Xsize = size(X); if (Xsize(1)~=1), error('The first parameter is not a row vector'); end
```

```
n = Xsize(2);
if (n<3), error('The number of the observations is too small'); end
if (nargin<3), alpha = 2; end for (i=1:n)
   x = X(i);
%  small 'x' is required in hypothetical cdf definition string F0
   Y(i) = eval(F0);
end Y = [0 sort(Y) 1]; dY = diff(Y(1:n)); if (abs(alpha)<eps)
% equation 7.12
   T = -sum(log(n*dY))-log(n*(1-Y(n)));
elseif (abs(1-alpha)<eps)
% equation 7.11
   T = log(n*dY)*(n*dY)'+log(n*(1-Y(n)))*n*(1-Y(n));
else
% equation 7.10
   T = (sum((n*dY).^alpha)+(n*(1-Y(n)))^alpha-n)/alpha/(alpha-1);
end return
```

**2.** Evaluation of the statistic $S_{\phi_\alpha,n}$ given by (7.13) - (7.15) and denoted as function S = Spdt.

```
function S = Spdt(X, F0, alpha)
% This function compute the Goodness-of-fit statistic based on
% the power-divergences.
%
% Use: Spdt(X, F0, {alpha})
%    x       row vector; observed data
%    F0      string: hypothetical cumulative distribution distribution of vector x
%    {alpha} number: from interval (-1/2, inf) order of power-divergence
%                    (optional, default = 2)
%
% Example: Spdt(X, 'normal_cdf(x, 0, 1)', 1.5)
eps = 1.E-4;
if (nargin<2), error('Use: Spdt(X, F0, {alpha})'); end
Xsize = size(X);
if (Xsize(1)~=1), error('The first parameter is not a row vector'); end
n = Xsize(2);
if (n<3), error('The number of the observations is too small'); end
if (nargin<3), alpha = 2; end
for (i=1:n)
   x = X(i);
%  small 'x' is required in hypothetical cdf definition string F0
   Y(i) = eval(F0);
end
Y = [0 sort(Y) 1];
dY = diff(Y);
if (abs(alpha)<eps)
% equation 7.15
   S = -sum(log((n+1)*dY));
elseif (abs(1-alpha)<eps)
% equation 7.14
   S = log((n+1)*dY)*((n+1)*dY)';
else
% equation 7.13
   S = (sum(((n+1)*dY).^alpha)-n-1)/alpha/(alpha-1);
end
return
```

**3.** Evaluation of the statistic $\tilde{T}_{\phi_\alpha, n}$ given by (7.16) - (7.18) and denoted as function TW = TWpdt.

```
function TW = TWpdt(X, F0, alpha)
% This function compute the Goodness-of-fit statistic based on
% the power-divergences.
%
% Use: TWpdt(X, F0, {alpha})
%    x       row vector; observed data
%    F0      string: hypothetical cumulative distribution distribution of vector x
%    {alpha} number: from interval (-1/2, inf) order of power-divergence
%                    (optional, default = 2)
%
% Example: TWpdt(X, 'normal_cdf(x, 0, 1)', 1.5)
eps = 1.E-4;
if (nargin<2), error('Use: TWpdt(X, F0, {alpha})'); end
Xsize = size(X);
if (Xsize(1)~=1), error('The first parameter is not a row vector'); end
n = Xsize(2);
if (n<3), error('The number of the observations is too small'); end
if (nargin<3), alpha = 2; end
for (i=1:n)
   x = X(i);
%  small 'x' is required in hypothetical cdf definition string F0
   Y(i) = eval(F0);
end
Y = [0 sort(Y) 1];
dY = diff(Y(2:n+1));
if (abs(alpha)<eps)
% equation 7.18
   TW = -sum(log(n*dY))-log(n*(Y(2)+1-Y(n+1)));
elseif (abs(1-alpha)<eps)
% equation 7.17
   TW = log(n*dY)*(n*dY)'+log(n*(Y(2)+1-Y(n+1)))*n*(Y(2)+1-Y(n+1));
else
% equation 7.16
   TW = (sum((n*dY).^alpha)+(n*(Y(2)+1-Y(n+1)))^alpha-n)/alpha/(alpha-1);
end
return
```

**Example 8.1**  Applicability of the PODIST package was verified by evaluating all three statistics in the domain $-1 < \alpha \le 3$ for the empirical distributions resulting from the Pearson crab measurement data of Table 1.1 and for the hypothetic distributions given by the mixed generalized lambda models MixGLD of Example 2.2.1. The problem faced here is that the data in this table are in fact discrete, obtained by quantization of the original continuous measurement outcomes. Due to this, the majority of the formally evaluated spacings is zero. These data can be brought into conformity with the assumptions of the present section by reconstructing the original nonquantized Pearson's data by a suitable random smoothing. We used the method of spreading the data from each quantization interval over all this interval uniformly randomly. The results for various realizations of this smoothing procedure were practically the same. They are presented in graphical form in Figure 8.1 where the values of statistic $T = T_{\phi_\alpha, n}$ are printed in green, the values of $S = S_{\phi_\alpha, n}$ are printed in red and the values of TW = $\tilde{T}_{\phi_\alpha, n}$ are printed in blue. Moreover, the asymptotic theory of Section 7 for the statistics from the class (7.4) was applied to evaluate the critical values as functions $-1 < \alpha \le 3$ for rejection of the MixGLD models at the 95% significance level. These values are printed in black dotted curves.The results are presented in graphical form in Figure 8.2 where the values of statistic $T = T_{\phi_\alpha, n}$ are printed in green, the values of $S = S_{\phi_\alpha, n}$ are printed in red and the values of TW = $\tilde{T}_{\phi_\alpha, n}$ are printed in blue. Similarly as in the previous example,

the asymptotic normality result of Assertion 7.3.1 was applied to evaluate the critical values as functions $-1 < \alpha \leq 3$ for rejection of the MixGLD models at the 95% significance level. These values are printed in black dotted curves. We see from Figure 8.1 that the differences between the statistics T $= T_{\phi_\alpha, n}$, S $= S_{\phi_\alpha, n}$ and TW $= \tilde{T}_{\phi_\alpha, n}$ are in the whole domain of powers $-1 < \alpha \leq 3$ small. This indicates that the asymptotic equivalence of Subsection 6.1 works well for the concrete sample size $n = 1000$ in the case of Table 1.1. On the other hand, we see from Figure 8.1 that all three continua of statistics presented there are in the rejection domain of all three statistics of the orders $0 < \alpha \leq 3$ and slightly inside this domain for the orders $-1 < \alpha < 0$. The 95% rejection domain is conservative in the sense that it guarantees at most 0.05 probability of the wrong rejection decision while the alternative non-rejection decision has no formal guarantee. This means that the statistics the positive orders find a sufficient stochastic argument to reject the hypothesis that the crab data under consideration were generated by the MixGLD model of Example 2.2.1 while the statistics of the negative orders do not go so far and admit (without a formal stochastic guarantee) this hypothesis. Similarly as in the comments to the tables of Section 2, one has take into account that the rejection decisions are based here on disparity between the hypothetical d.f. $F$ and the empirical d.f. $F_n$. This does not exclude the possibility that the fit between model and data is decided differently by the criteria taking into account some local properties of $F$ and $F_n$.
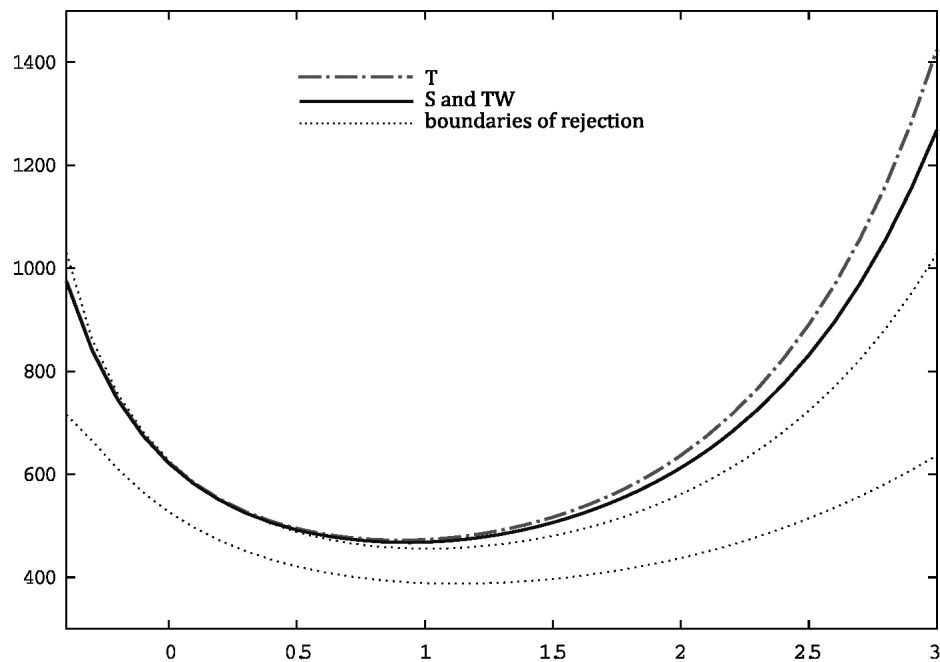


Figure 8.1: Statistics T $= T_{\phi_\alpha, n}$, S $= S_{\phi_\alpha, n}$ and TW $= \tilde{T}_{\phi_\alpha, n}$ for $-1/2 < \alpha \leq 3$ and the data and MixGLD model of Example 2.2.1.

We have seen in the last example that the infinite families of statistics introduced in this chapter provides broader picture of agreement or disagreement between statistical model and data than just one such statistic, thus creating a safer basis for the final statistician's decision. At the same time we have seen that the continuum of statistics creates a new problem for him, namely to weight eventual differences between recommendations resulting from different subclasses of statistics. However, sometimes it is not so, as it is illustrated by the next example.

**Example 8.2** The PODISTAT package was verified also by evaluating its three families of statistics in the domain $-1 < \alpha \leq 3$ for the empirical distributions resulting from the horse kidney measurement data of Table 1.2 and for the hypothetical distributions given by the mixed generalized lambda models

MixGLD of Example 2.2.2. The results are presented in graphical form in Figure 8.2 where the values of statistic T $= T_{\phi_\alpha,n}$ are printed in green, the values of S $= S_{\phi_\alpha,n}$ are printed in red and the values of TW $= \tilde{T}_{\phi_\alpha,n}$ are printed in blue. Similarly as in the previous example, the asymptotic normality result of Assertion 7.3.1 was applied to evaluate the critical values as functions $-1/2 < \alpha \leq 3$ for rejection of the MixGLD models at the 95% significance level. These values are printed in black dotted curves. We see from Figure 8.2 that, similarly as in the previous example, the differences between the statistics T $= T_{\phi_\alpha,n}$, S $= S_{\phi_\alpha,n}$ and TW $= \tilde{T}_{\phi_\alpha,n}$ are negligible for all powers $-1 < \alpha \leq 3$ which indicates that the asymptotic equivalence of Subsection 6.1 works already for moderate sample sizes like $n = 43$ in the case of Table 1.2. Further we see from Figure 8.2 that the MixGLD of Example 2.2.2 is not rejected at the 95% significance level. The conservative character of the hypothesis rejection rule mentioned in Example 8.1 is manifested here by much wider non-rejection domain corresponding to the small sample size $n = 43$ than the domain corresponding to the large sample size $n = 1000$ in Example 8.1. All statistics under consideration indicate that the MixGLD of Example 2.2.2 is well fitted to the horse kidney data in the sense that all of them are localized deeply near the middle of the non-rejection domain.
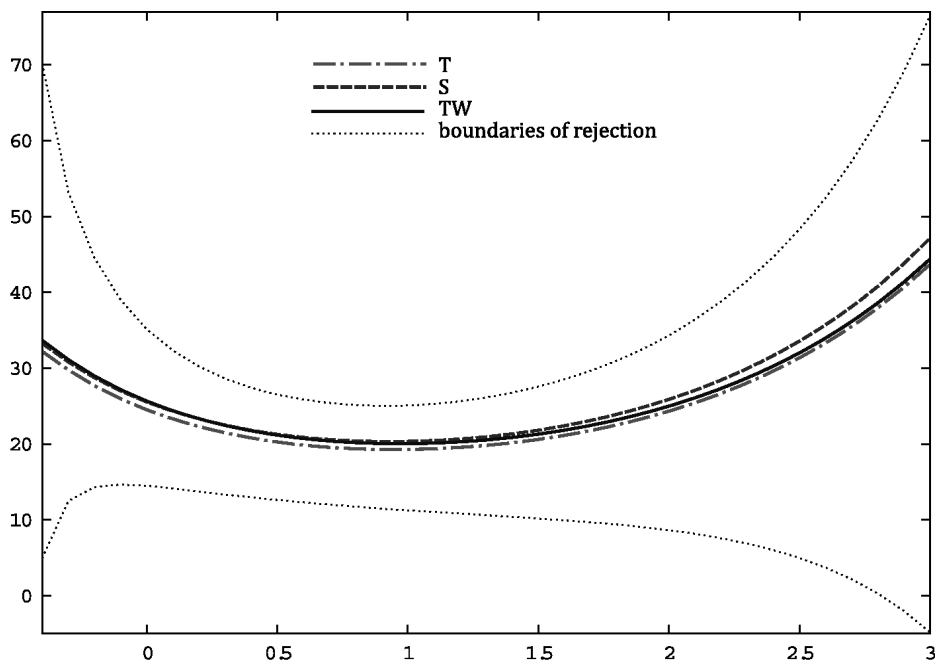


Figure 8.2: Statistics T $= T_{\phi_\alpha,n}$, S $= S_{\phi_\alpha,n}$ and TW $= \tilde{T}_{\phi_\alpha,n}$ for $-1/2 < \alpha \leq 3$ and the data and MixGLD model of Example 2.2.2.

# 9 Appendix

Proofs of the assertions stated in this chapter above can be found in Vajda and van der Meulen (2010). They are presented here for the sake of completeness.

## 9.1 Proofs for structural spacings statistics

**Proof of Assertion 6.1.1** We shall consider the fixed alternative $F(x)$ with a continuous density $f(x) > 0$ for $0 \leq x \leq 1$. For the local alternatives the argument is similar. By inspecting the definitions

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of $T_\phi$, $\tilde{T}_\phi$ and $R_\phi$ we see that for (6.3) it suffices to prove that as $n \to \infty$

$$\phi(np_{01}) = O_p(1) \quad \text{and} \quad \phi(n(p_{01} + p_{02})) = O_p(1). \tag{9.50}$$

It is known (see for example page 208 in Hall (1986)) that $p_{01} = F^{-1}(Z_1/W_{n+1})$ and $p_{01} + p_{02} = F^{-1}((Z_1 + Z_2)/W_{n+1})$, where $Z_1, \ldots, Z_{n+1}$ are independent standard exponential variables and $W_{n+1} = Z_1 + \cdots + Z_{n+1}$, so that, for $n \to \infty$,

$$\frac{W_{n+1}}{n} \xrightarrow{p} 1 \quad \text{and} \quad V_n = \frac{Z_1}{W_{n+1}} \xrightarrow{p} 0.$$

Setting

$$\Upsilon_n = \frac{F^{-1}(V_n)}{V_n} = \frac{F^{-1}(V_n) - F^{-1}(0)}{V_n}$$

and using the mean value theorem and the assumed continuity of $f$ in the neighborhood of 0, we find that

$$\Upsilon_n \xrightarrow{p} \frac{1}{f(0)} \quad \text{as } n \to \infty$$

where, by assumptions about $f$, $0 < f(0) < \infty$. Thus

$$np_{01} = \frac{n}{W_{n+1}} Z_1 \, \Upsilon_n$$

and, by applying (5.38),

$$\phi(np_{01}) = \xi\left(\frac{n}{W_{n+1}}\right) \phi(Z_1 \, \Upsilon_n) + \zeta(Z_1 \, \Upsilon_n) \, \phi\left(\frac{n}{W_{n+1}}\right) + \eta\left(\frac{n}{W_{n+1}}\right) (Z_1 \, \Upsilon_n - 1).$$

Since $Z_1 \, \Upsilon_n = O_p(1)$ as $n \to \infty$, we obtain from Assertion 5.3.1

$$
\begin{aligned}
\phi(np_{01}) &= \left[ \xi\left(\frac{n}{W_{n+1}}\right) + \phi\left(\frac{n}{W_{n+1}}\right) + \eta\left(\frac{n}{W_{n+1}}\right) \right] O_p(1) \\
&= [\xi(1) + \phi(1) + \eta(1) + o_p(1)] \, O_p(1) \\
&= O_p(1) \quad (\text{cf } (5.40)),
\end{aligned}
$$

thus proving the first relation of (9.50). Replacing $V_n = Z_1/W_{n+1}$ by $V_n = (Z_1 + Z_2)/W_{n+1}$, and using the fact that now

$$(Z_1 + Z_2) \, \Upsilon_n = (Z_1 + Z_2) \frac{F^{-1}(V_n) - F^{-1}(0)}{V_n} = O_p(1)$$

we obtain the second relation of (9.50). Next we prove (6.4). From (5.39) we get for any $p > 0$

$$\phi((n+1)\, p) = \xi\left(\frac{n+1}{n}\right) \phi(np) + \phi\left(\frac{n+1}{n}\right) + \eta\left(\frac{n+1}{n}\right) (np - 1)$$

so that

$$\phi((n+1)\, p) - \phi(np) = \varepsilon_n \phi(np) + \phi\left(\frac{n+1}{n}\right) + \eta\left(\frac{n+1}{n}\right) (np - 1) \tag{9.51}$$

where $\varepsilon_n = \xi((n+1)/n) - 1 = o(1)$ as $n \to \infty$ by Assertion 5.3.1. Replacing $p$ by the probabilities $p_{0j} = Y_j - Y_{j-1}$ figuring in the definitions of $S_\phi$ and $R_\phi$ (cf. (5.2) and (5.1)), and summing over $1 \le j \le n+1$, we get the equality

$$S_\phi - R_\phi = \varepsilon_n \, R_\phi + \delta_n$$

for

$$
\begin{aligned}
\delta_n &= (n+1)\, \phi\left(\frac{n+1}{n}\right) - \eta\left(\frac{n+1}{n}\right) \\
&= \frac{n+1}{n} \frac{\phi\left(1 + \frac{1}{n}\right) - \phi(1)}{\frac{1}{n}} - \eta\left(\frac{n+1}{n}\right).
\end{aligned}
$$

By Assertion 5.3.1,

$$\delta_n = \phi'(1) + o(1) \quad \text{as } n \to \infty.$$

This completes the proof of the first relation in (6.4). The proof of the second relation is the same: we just replace $p$ in (9.51) by the probabilities $\tilde{p}_{0j}$ figuring in the definition (5.22) of $\tilde{T}_\phi$. $\qquad \square$

**Proof of Assertion 6.3.1** By Theorem 1 of Hall (1984), the statistic $\tilde{R}_\phi$ defined by (5.25) satisfies under a fixed alternative $F \sim f$ the relation

$$\frac{\tilde{R}_\phi}{n} \xrightarrow{p} \tilde{\mu}_\phi(f) = \int_0^1 f^2(x) \left( \int_0^\infty \phi(t)\, e^{-tf(x)} \mathrm{d}t \right) \mathrm{d}x \quad \text{as } n \to \infty$$

provided $\phi : (0, \infty) \mapsto \mathbb{R}$ is continuous and exponentially bounded in the sense that $|\phi(t)| \le K(t^\alpha + t^{-\beta})$ for some $K > 0$, $\alpha \ge 0$, $\beta < 1$, and $f$ is bounded, piecewise continuous, and bounded away from $0$ (see also part (i) of Theorem 3.1 in Misra and van der Meulen (2001)). Thus (6.14) is proved for $U_\phi = \tilde{R}_\phi$ as soon as it is shown that for $\phi \in \mathbf{\Phi}_1$ the limit $\tilde{\mu}_\phi(f)$ coincides with $\mu_\phi(f)$. By substituting $s$ for $tf(x)$ in the last integral, and using the assumption $0 < f(x) < \infty$ and the functional equation (5.38),

$$
\begin{aligned}
\tilde{\mu}_\phi(f) &= \int_0^1 f(x) \left( \int_0^\infty \phi\left( \frac{s}{f(x)} \right) e^{-s} \mathrm{d}s \right) \mathrm{d}x & (9.52) \\
&= \int_0^1 f(x) \left( \int_0^\infty \left[ \xi(s)\, \phi\left( \frac{1}{f(x)} \right) + \zeta\left( \frac{1}{f(x)} \right) \phi(s) + \eta(s) \left( \frac{1}{f(x)} - 1 \right) \right] e^{-s} \mathrm{d}s \right) \mathrm{d}x \\
&= \mu_\phi(f) + \int_0^\infty \eta(s)\, e^{-s}\, \mathrm{d}s \int_0^1 (1 - f(x))\, \mathrm{d}x = \mu_\phi(f).
\end{aligned}
$$

The extension of (6.14) to $U_\phi \in \{T_\phi, \tilde{T}_\phi, R_\phi\}$ follows from Assertion 6.1.1. For $\phi \in \mathbf{\Phi}_2$ the extension of (6.14) to $U_\phi \in \{S_\phi, \tilde{S}_\phi\}$ follows from Assertion 6.1.1 too. $\qquad\square$

**Proof of Assertion 6.4.1** For $U_\phi = S_\phi$ the relations (6.16)–(6.18) follow from the result of Kuo and Rao (1981), cf. also Del Pino (1979) and Theorem 3.2 in Misra and van der Meulen (2001). The extension to the remaining statistics $U_\phi$ follows from Assertion 6.1.1. $\qquad\square$

**Proof of Assertion 6.5.1** Consider $U_\phi = \tilde{R}_\phi$ for $\phi \in \mathbf{\Phi}_2$. By Assertion 5.3.2, $\phi(t)$ has a continuous derivative $\phi'(t)$ on $(0, \infty)$. By (5.41), for every $c \in \mathbb{R}$

$$t^c |\phi'(t)| \le |\xi'(1)|\, t^{c-1} |\phi(t)| + |\phi'(1)|\, t^c + |\eta'(1)|\, t^{c-1} |t - 1|.$$

Thus if $\phi$ satisfies (6.5) with $\beta < 1/2$ then there exists $\alpha \ge 0$ such that

$$\lim_{t \to \infty} t^{-\alpha} |\phi'(t)| = \lim_{t \downarrow 0} t^{1+\beta} |\phi'(t)| = 0.$$

This means that under the assumptions of Assertion 6.4.1 there exist $c > 0$, $K > 0$ and $b < 1/2$ such that for every $t \in (0, \infty)$

$$|\phi(t)| \le K(t^a + t^{-b}) \quad \text{and} \quad |\phi'(t)| \le K(t^a + t^{-b-1}).$$

For continuously differentiable functions $\phi$ satisfying these assumptions, and fixed alternatives with densities $f$ continuously differentiable on $(0, 1)$, it follows from Theorem 2 in Hall (1984) (cf. also part (ii) of Theorem 3.1 in Misra and van der Meulen (2001)) that $U_\phi = \tilde{R}_\phi$ satisfies the relation

$$\frac{1}{\sqrt{n}} (U_\phi - n\tilde{\mu}_\phi(f)) \xrightarrow{\mathcal{D}} N(0, \tilde{\sigma}_\phi^2(f)) \quad \text{for } n \to \infty$$

where: **(1)** the asymptotic mean $\tilde{\mu}_\phi(f)$ was presented and proved to be equal to $\mu_\phi(f)$ in the proof of Assertion 6.3.1 under assumptions weaker than here and, **(2)** the asymptotic variance $\tilde{\sigma}_\phi^2(f)$ can be specified by means of the standard exponential variable $Z$ and the auxiliary function

$$G(x) = \int_0^x \left( 1 - \frac{F(y)\, f'(y)}{f^2(y)} \right) E\left[ Z\, \phi'\left( \frac{Z}{f(y)} \right) \right] \mathrm{d}y, \quad 0 < x < 1, \qquad (9.53)$$

as the sum of

$$s_1^2(f) \quad = \quad \int_0^1 \left( E\phi^2 \left( \frac{Z}{f(x)} \right) - \left[ E\phi \left( \frac{Z}{f(x)} \right) \right]^2 \right) f(x)\, dx \tag{9.54}$$

$$s_2^2(f) \quad = \quad -2 \int_0^1 E \left[ (Z-1)\, \phi \left( \frac{Z}{f(x)} \right) \right] \frac{G(x)}{F(x)}\, f(x)\, dx \tag{9.55}$$

and

$$s_3^2(f) = \int_0^1 \left( \frac{G(x)}{F(x)} \right)^2 f(x)\, dx. \tag{9.56}$$

It remains to be proved that for every $x \in (0,1)$

$$\left( E\, \phi^2 \left( \frac{Z}{f(x)} \right) - \left[ E\, \phi \left( \frac{Z}{f(x)} \right) \right]^2 \right) f(x) = \Psi_2(x), \tag{9.57}$$

$$E \left[ (Z-1)\, \phi \left( \frac{Z}{f(x)} \right) \right] \sqrt{f(x)} = \Psi_3(x) \tag{9.58}$$

and

$$\frac{G(x)\, \sqrt{f(x)}}{F(x)} = \Psi_4(x). \tag{9.59}$$

Indeed, then $\tilde{\sigma}_\phi^2(t) = \sigma_\phi^2(f)$ so that (6.23) is proved for $U_\phi = R_\phi$, and the extension of (6.23) to the remaining statistics $U_\phi \in \{\tilde{R}_\phi, S_\phi, \tilde{S}_\phi, T_\phi, \tilde{T}_\phi\}$ follows from Assertion 6.1.1. We shall prove (9.57)–(9.59) in the reversed order. By substituting $t = Z/f(y)$ in (5.41) and taking into account that $\zeta(t) = 1$ we obtain

$$
\begin{aligned}
E \left[ Z\phi' \left( \frac{Z}{f(y)} \right) \right] \quad &= \quad f(y)\, E \left[ \xi'(1)\, \phi \left( \frac{Z}{f(y)} \right) + \phi'(1) + \eta'(1) \left( \frac{Z}{f(y)} - 1 \right) \right] \\
&= \quad f(y) \left[ \xi'(1)\, E\, \phi \left( \frac{Z}{f(y)} \right) + \phi'(1) + \eta'(1) \left( \frac{1}{f(y)} - 1 \right) \right]
\end{aligned}
$$

and, by putting $s = 1/f(x)$ and $t = Z$ in (5.39), we get

$$\phi \left( \frac{Z}{f(x)} \right) = \phi(Z)\, \xi \left( \frac{1}{f(x)} \right) + \phi \left( \frac{1}{f(x)} \right) + \eta \left( \frac{1}{f(x)} \right) (Z-1). \tag{9.60}$$

Therefore

$$E\, \phi \left( \frac{Z}{f(x)} \right) = \langle \phi \rangle \xi \left( \frac{1}{f(x)} \right) + \phi \left( \frac{1}{f(x)} \right) \tag{9.61}$$

and, consequently,

$$E \left[ Z\, \phi' \left( \frac{Z}{f(y)} \right) \right] = \Psi_1(y). \tag{9.62}$$

This, together with the definitions of $\Psi_4(x)$ and $G(x)$ in (6.22) and (9.53), implies (9.59). Further, from (9.60) and the definition of $\Psi_3(x)$ in (6.21) we get (9.58). Finally, from (9.60), (9.61) and the definition of $\Psi_2(x)$ in (6.20) we obtain (9.57) which completes the proof. $\qquad \square$

## 9.2   Proofs for power spacings statistics

**Proof of Assertion 7.1.1**   By the assumptions about $g$,

$$t_0 = \min_{y \in [a,b]} g(y) > 0 \quad \text{and} \quad t_1 = \max_{y \in [a,b]} g(y) < \infty.$$

If $\psi_\alpha(t)$ is convex, then for every $t \in [t_0, t_1]$ and $\alpha \in (c,d)$

$$\psi_\alpha'(t_*)\, (t - t_*) \le \psi_\alpha(t) \le \psi_\alpha(t_0) + \psi_\alpha(t_1).$$

If $\psi_\alpha(t)$ is concave, then, similarly,

$$\psi_\alpha(t_0) + \psi_\alpha(t_1) \le \psi_\alpha(t) \le \psi'_\alpha(t_*)\,(t - t_*).$$

Therefore in both cases

$$\max_{t_0 \le t \le t_1} |\psi_\alpha(t)| \le \max\left\{|\psi_\alpha(t_0) + \psi_\alpha(t_1)|,\ |\psi'_\alpha(t_*)| \cdot |t_1 - t_0|\right\}.$$

The assumed continuity of $\psi'_\alpha(t_*)$ and $\psi_\alpha(t_0) + \psi_\alpha(t_1)$ in the variable $\alpha \in (c, d)$ implies that for all compact neighborhoods $N \subset (c, d)$ of $\alpha_0$ the constant

$$k = \sup_{\alpha \in N} \max_{t_0 \le t \le t_1} |\psi_\alpha(t)| = \sup_{\alpha \in N} \max_{y \in [a,b]} |\psi_\alpha(g(y))|$$

is finite. Put

$$K = \max_{[t_0,t_1] \times [-k,k]} \Phi(u, v).$$

The function $|\Phi(g, \psi_\alpha(g))|$ of variables $(y, \alpha) \in [a, b] \times (c, d)$ is bounded on $[a, b] \times N$ by $K < \infty$. Since for every $y \in [a, b]$

$$\lim_{\alpha \to \alpha_0} \Phi(g, \psi_\alpha(g)) = \Phi(g, \psi_{\alpha_0}(g)),$$

the Lebesgue dominated convergence theorem for integrals implies (7.21). $\qquad\square$

**Proof of Assertion 7.1.2** Since $\mu_\alpha = \mu_\alpha(f_0)$ and $\sigma_\alpha^2 = \sigma_\alpha^2(f_0)$, where the hypothetical density $f_0$ satisfies the assumptions of Assertions 6.3.1 and 6.5.1, the continuity of $\mu_\alpha$ and $\sigma_\alpha^2$ follows from the continuity of $\mu_\alpha(f)$ and $\sigma_\alpha^2(f)$ proved below. By (7.20) and (6.18),

$$m_\alpha(\ell) = \frac{\|\ell\|^2}{2}\left(\langle t^2 \phi_\alpha(t)\rangle - 4\langle t\phi_\alpha(t)\rangle + 2\langle \phi_\alpha(t)\rangle\right)$$

where $\phi_\alpha$ is given by (3.26), (3.27), and, by (6.6),

$$\langle t^j \phi_\alpha(t)\rangle = \int_0^\infty t^j \phi_\alpha(t)\,\mathrm{d}H(t), \quad j \in \{0, 1, 2\} \tag{9.63}$$

for $H(t) = 1 - e^{-t}$. All integrals (9.63) are finite if and only if $\alpha \in (-1, \infty)$. Further, for every fixed $t > 0$

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\,\alpha\phi_\alpha(t) \ge 0 \quad \text{at any } \alpha \in \mathbb{R}. \tag{9.64}$$

Hence the continuity of the products $\alpha\langle t^j \phi_\alpha(t)\rangle$ in the variable $\alpha \in \mathbb{R}$ follows from the monotone convergence theorem for integrals, and this implies also the desired continuity of the integrals (9.63) at any $\alpha \in (-1, \infty) - \{0\}$. Further, for every fixed $t > 0$

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\,(\alpha - 1)\,\phi_\alpha(t) \ge 0 \quad \text{for any } \alpha \in \mathbb{R}. \tag{9.65}$$

Hence the continuity of the products $(\alpha - 1)\,\langle t^j \phi_\alpha(t)\rangle$ in the variable $\alpha \in \mathbb{R}$ follows as well from the monotone convergence theorem for integrals. Similarly as above, this implies the continuity of the integrals (9.63) at the remaining point $\alpha = 0$. Further, by (7.20) and (6.12),

$$\mu_\alpha(f) = \langle \xi_\alpha\rangle\, D_\alpha(F_0, F) + \langle \phi_\alpha\rangle$$

where, by (6.6) and (7.5)

$$\langle \xi_\alpha\rangle = \int_0^\infty t^\alpha \mathrm{d}H(t) \quad \text{and} \quad \langle \phi_\alpha\rangle = \int_0^\infty \phi_\alpha(t)\,\mathrm{d}H(t).$$

These integrals are finite if and only if $\alpha \in (-1, \infty)$. The continuity of $\langle \phi_\alpha\rangle$ at $\alpha \in (-1, \infty)$ was proved above, the continuity of $D_\alpha(F_0, F)$ at $\alpha \in \mathbb{R}$ follows from the assumptions about the densities $f_0$ and $f$

and from Proposition 2.14 in Liese and Vajda (1987). The continuity of $\langle\xi_\alpha\rangle$ at $\alpha \in (-1, \infty)$ follows from the monotone convergence theorem for integrals applied separately to the integration domains $(0, 1)$ and $(1, \infty)$. Finally, let us consider $\sigma_\alpha^2(f)$ defined by $(6.19)$–$(6.24)$ for $\phi = \phi_\alpha$, $\xi = \xi_\alpha$, and $\eta = \eta_\alpha$ given by $(3.26)$, $(3.27)$ and $(7.5)$. The integrals $\langle t\phi_\alpha(t)\rangle$, $\langle\phi_\alpha(t)\rangle$ and $\langle\phi_\alpha^2(t)\rangle$ are finite if and only if $\alpha \in (-1/2, \infty)$, and their continuity at $\alpha \in (-1/2, \infty)$ was either proved above or can be proved similarly as above. The continuity of the integral

$$\int_0^1 \left[ f\xi_\alpha^2\left(\frac{1}{f}\right) + f\eta_\alpha^2\left(\frac{1}{f}\right) \right] dx$$

at $\alpha \in (-1/2, \infty)$ follows from Assertion 7.1.1, which establishes the continuity of the component $\int \Psi_2(x)\, dx$ of $\sigma_\alpha^2(f)$ in $(6.24)$. For the continuity of the remaining two components, we take into account that $F(x) > c_1 x$ for some $c_1 > 0$ on $[0, 1]$, because $f$ is bounded away from zero on $[0, 1]$. Furthermore, both $f(x)$ and $f'(x)$ are bounded on $[0, 1]$, so that there exists a constant $c_2$ such that in $(6.22)$

$$\frac{\sqrt{f(x)}}{F(x)} \int_0^x \left| 1 - \frac{F(y)\, f'(y)}{f^2(y)} \right| dy < c_2 \quad \text{for all } x \in [0, 1]. \tag{9.66}$$

Using the function $\varphi_\alpha(t) = \alpha\phi_\alpha(t)$, which is for every $t > 0$ continuous and monotone in $\alpha \in \mathbb{R}$ (cf. $(9.64)$), we obtain from $(6.19)$

$$\Psi_1(x) = \alpha\langle\phi_\alpha\rangle\, f(x)^{1-\alpha} + f(x)\, \varphi_\alpha\left(\frac{1}{f(x)}\right) + 1 - f(x)$$

where the right-hand side is bounded on $[0, 1]$, locally uniformly in $\alpha$, and continuous at any $\alpha \in \mathbb{R}$. By $(6.22)$ and $(9.66)$, this implies that also $\Psi_4(x)$ is bounded on $[0, 1]$, locally uniformly in $\alpha$, and continuous at any $\alpha \in \mathbb{R}$. Since the integrands in

$$\int_0^1 \left[ \sqrt{f}\xi_\alpha\left(\frac{1}{f}\right) + \sqrt{f}\eta_\alpha\left(\frac{1}{f}\right) \right] \Psi_4\, dx \quad \text{and} \quad \int_0^1 \Psi_4^2\, dx$$

are continuous on $[0, 1]$ and locally bounded in the variable $\alpha \in \mathbb{R}$, the continuity of both these integrals in the variable $\alpha \in \mathbb{R}$ follows from the Lebesgue dominated convergence theorem for integrals. This clarifies the continuity of the second and third component of $\sigma_\alpha^2(f)$ in $(6.24)$ and thus completes the proof. $\qquad\square$

**Proof of Assertion 7.2.1**  The functions from the class $\{\phi_\alpha : \alpha \in (-1, \infty)\} \subset \mathbf{\Phi}_2$ satisfy all assumptions of Assertion 6.3.1. Hence $(7.23)$ holds for all $\alpha > -1$ and the limit $\mu_\alpha(f)$ is given in accordance with $(6.12)$ and $(7.5)$ by the formula

$$\mu_\alpha(f) = \langle\xi_\alpha(t)\rangle\, D_\alpha(F_0, F) + \langle\phi_\alpha(t)\rangle = \langle t^\alpha\rangle\, D_\alpha(F_0, F) + \langle\phi_\alpha(t)\rangle$$

where $\langle t^\alpha\rangle = \Gamma(\alpha + 1)$ for all $\alpha \in \mathbb{R}$. If $\alpha \notin \{0, 1\}$ then

$$\langle\phi_\alpha(t)\rangle = \frac{1}{\alpha(\alpha - 1)}\langle t^\alpha - 1\rangle = \frac{\Gamma(\alpha + 1) - \Gamma(1)}{\alpha(\alpha - 1)}$$

but the expressions

$$\langle\phi_0(t)\rangle = \langle -\ln t\rangle \quad \text{and} \quad \langle\phi_1(t)\rangle = \langle t\ln t\rangle$$

lead to the evaluation of unpleasant integrals. This evaluation can be avoided by employing Assertion 7.1.2. From the continuity of $\mu_\alpha = \langle\phi_\alpha(t)\rangle$, it follows that

$$\mu_j = \langle\phi_j(t)\rangle = \lim_{\alpha \to j} \frac{\Gamma(\alpha + 1) - \Gamma(1)}{\alpha(\alpha - 1)} \quad \text{for } j \in \{0, 1\},$$

where the limit on the right can be easily evaluated by using L'Hospital's rule and the known formulas $\Gamma'(1) = -\gamma$, $\Gamma'(2) = 1 - \gamma$, thus leading to the values $\mu_j$, $j \in \{0, 1\}$, given in $(7.25)$. The continuity and the inequality $\mu_\alpha(f) \geq \mu_\alpha$ for $\alpha \in (-1, \infty)$ follow from $(7.24)$ and $(7.25)$ because $D_\alpha(F_0, F)$ is nonnegative and continuous in $\alpha \in \mathbb{R}$ and $\Gamma(\alpha + 1)$ is positive and continuous in $\alpha \in (-1, \infty)$. The condition for equality follows from the fact that $D_\alpha(F_0, F)$ is positive unless $F = F_0$. $\qquad\square$

**Proof of Assertion 7.3.1** Similarly as we applied Assertion 6.3.1 in the proof of Assertion 7.2.1, (7.28) follows for all $\alpha > -1/2$ from Assertion 6.4.1. If $\alpha \notin \{0,1\}$, then the expressions for $m_\alpha(\ell)$ and $\sigma_\alpha^2$ given in (7.29) and (7.30) follow easily from the formulas given for $m_{\phi_\alpha}(\ell)$ and $\sigma_{\phi_\alpha}^2$ in Assertion 6.4.1, but the direct evaluation of $m_j(\ell)$ and $\sigma_j^2$ from these formulas for $j \in \{0,1\}$ is a somewhat tedious task. However, by using the continuity of $m_\alpha(\ell)$ and $\sigma_\alpha^2$ established in Assertion 7.1.2, we obtain $m_j(\ell)$ and $\sigma_j^2$ given in (7.29) and (7.31) as the limits

$$m_j(\ell) = \lim_{\alpha \to j} m_\alpha(\ell) \quad \text{and} \quad \sigma_j^2 = \lim_{\alpha \to j} \sigma_\alpha^2 \quad \text{for } j \in \{0,1\},$$

which expressions can be easily evaluated by using the continuity of the right-hand side of (7.29) and L'Hospital's rule, thereby employing the formulas

$$
\begin{aligned}
\Gamma(\alpha + k + 1) &= (\alpha + k)(\alpha + k - 1) \cdots (\alpha + 1)\,\Gamma(\alpha + 1), \\
\Gamma''(\alpha + 1) &= 2\Gamma'(\alpha) + \alpha\Gamma''(\alpha)
\end{aligned}
$$

and

$$\Gamma''(1) = \frac{\pi^2}{6} + \gamma^2, \quad \Gamma''(2) = \frac{\pi^2}{6} - 2\gamma + \gamma^2, \quad \Gamma''(3) = \frac{\pi^2}{3} + 2 - 6\gamma + 2\gamma^2$$

in addition to the previously used $\Gamma'(1) = -\gamma$ and $\Gamma'(2) = 1 - \gamma$. $\qquad\square$

**Proof of Assertion 7.4.1** If $\psi : [0,1] \mapsto \mathbb{R}$ is continuous then by the assumptions about $f$

$$\inf_{x \in [0,1]} f(x) > 0 \quad \text{and} \quad \sup_{x \in [0,1]} |\psi(x)\,f(x)| < \infty$$

and, consequently, the function

$$\Psi(x) = \int_0^x \psi(y)\,f(y)\,\mathrm{d}y, \quad x \in (0,1)$$

is well defined. Since

$$\frac{\mathrm{d}}{\mathrm{d}x} \frac{\Psi^2}{F} = -\left(\frac{\Psi}{F}\right)^2 f + \frac{2\Psi\psi f}{F}$$

and

$$|\Psi(y)| \le y \sup_{x \in [0,1]} |\psi(x)\,f(x)| \quad \text{as well as} \quad F(y) \ge y \inf_{x \in [0,1]} f(x),$$

the function $\Psi$ satisfies the relation

$$\int_0^1 (\psi - \Psi/F)^2 f\,\mathrm{d}x = \int_0^1 \psi^2 f\,\mathrm{d}x - \left(\int_0^1 \psi\,f\,\mathrm{d}x\right)^2. \tag{9.67}$$

To this end take into account the relations

$$
\begin{aligned}
\int_0^1 (\psi - \Psi/F)^2 f\,\mathrm{d}x &= \int_0^1 \psi^2 f\,\mathrm{d}x - \int_0^1 \frac{2\Psi\psi f}{F}\,\mathrm{d}x + \int_0^1 \left(\frac{\Psi}{F}\right)^2 f\,\mathrm{d}x \\
&= \int_0^1 \psi^2 f\,\mathrm{d}x - \left(\frac{\Psi^2(1)}{F(1)} - \lim_{y \downarrow 0} \frac{\Psi^2(y)}{F(y)}\right) \\
&= \int_0^1 \psi^2 f\,\mathrm{d}x - \frac{\Psi^2(1)}{F(1)}.
\end{aligned}
$$

Now, using (9.67) we obtain (7.39) from the definition (7.36). Since $f$ is assumed to be bounded and bounded away from 0,

$$
\begin{aligned}
\lim_{\alpha \to 0} \Delta_\alpha(F_0, F) &= \int_0^1 \left(\lim_{\alpha \to 0} \frac{f^{-\alpha} - 1}{\alpha} - \int_0^1 \lim_{\alpha \to 0} \frac{f^{-\alpha} - 1}{\alpha} f\,\mathrm{d}y\right)^2 f\,\mathrm{d}x \\
&= \int_0^1 \left(\ln f - \int_0^1 (\ln f)\,f\,\mathrm{d}y\right)^2 f\,\mathrm{d}x \\
&= \Delta_0(F_0, F)
\end{aligned}
$$

which proves the continuity at $\alpha = 0$. The reflexivity is clear from (7.39) and (7.40). $\qquad\square$

**Proof of Assertion 7.4.2** The proof should be clear from what was said above. The inequality $\sigma_\alpha^2(f) \geq \sigma_\alpha^2$ and the condition for equality follow from (7.38), because $D_{2\alpha}(F_0, F)$ and $\Delta_\alpha(F_0, F)$ are nonnegative measures of divergence of $F_0$ and $F$, which are equal to zero if and only if $F = F_0$, in which case the excess function (7.41) is 0. $\qquad\square$

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